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# Criteria for well-posedness of degenerate elliptic and parabolic problems

Maria Assunta Pozio *, Fabio Punzo, Alberto Tesei

*Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", P.le A. Moro 5, I-00185 Roma, Italy*

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**Abstract**

We provide explicit criteria for uniqueness or nonuniqueness of solutions to a wide class of second order elliptic and parabolic problems. The operator coefficients may be unbounded or vanish, or not to have a limit when approaching some part of the boundary, referred to as *singular boundary*. We discuss whether boundary conditions should be imposed on such a part to ensure well-posedness. The answer depends on the dimension of the singular boundary, and possibly on the behavior of coefficients near it.


**Résumé**

On donne des critères pour l'unicité ou la non unicité pour une importante classe de problèmes elliptiques et paraboliques du deuxième ordre. Les coefficients de l'opérateur différentiel peuvent être non bornés ou s'annuler, ou même ne pas avoir de limite lorsqu'on approche une partie de la frontière, désignée sous le nom de *frontière singulière*. La question se pose de savoir, si des conditions au bord doivent être imposées sur cette partie pour assurer que le problème soit bien posé. La réponse dépend autant de la dimension de la frontière singulière que du comportement des coefficients de l'opérateur au voisinage de cette frontière.


*MSC:* 35J25; 35K20

*Keywords:* Singular elliptic and parabolic problems; Well-posedness; Upper and lower solutions

## 1. Introduction

We study existence and uniqueness of solutions to linear *degenerate* elliptic equations of the following form:

$$\mathcal{L}u - cu = \phi \quad \text{in } \Omega. \tag{1.1}$$

Here $\Omega \subseteq \mathbb{R}^n$ is an open connected bounded set with boundary $\partial\Omega$ and $c, \phi$ are given functions, $c \geqslant 0$ in $\Omega$; the operator $\mathcal{L}$ is formally defined as follows:

* Corresponding author.
*E-mail addresses:* pozio@mat.uniroma1.it (M.A. Pozio), punzo@mat.uniroma1.it (F. Punzo), tesei@mat.uniroma1.it (A. Tesei).

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$$\mathcal{L}u \equiv \frac{1}{\rho}\mathcal{M}u \equiv \frac{1}{\rho(x)}\left[\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}\right].$$

We assume $\rho > 0$ in $\Omega$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant 0 \quad \text{for any } x \in \Omega,\ (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

Precise assumptions on the coefficients of Eq. (1.1) are made below (see assumptions $(A_2)$, $(E_1)$ and $(E_2)$).

Our methods also apply to companion parabolic equations of the form:

$$\mathcal{L}u - cu - \partial_t u = f \quad \text{in } \Omega \times (0, T) =: Q_T \tag{1.2}$$

with $T > 0$, $f = f(x,t)$ given; no sign condition on $c = c(x)$ is needed in this case (see Section 3). Quasilinear parabolic equations can be dealt with similarly (see [11,12]).

We always regard the boundary $\partial\Omega$ as the disjoint union of the *regular boundary* $\mathcal{R}$ and the *singular boundary* $\mathcal{S}$. The case $\partial\Omega \neq \partial\overline{\Omega}$ is possible, thus $\mathcal{S}$ can be a manifold of dimension less than $(n-1)$ (while $\mathcal{R} \subseteq \partial\overline{\Omega}$; see assumption $(A_1)$). In general, the coefficients of $\mathcal{L}$ and the function $c$ can either vanish or diverge, or need not have a limit, when $\operatorname{dist}(x, \mathcal{S}) \to 0$; moreover, ellipticity is possibly lost in $\Omega$ and/or when $\operatorname{dist}(x, \mathcal{S}) \to 0$ (see $(A_2)$, $(E_1)$ and $(E_2)$). Then it is natural to prescribe the Dirichlet boundary condition on the regular boundary $\mathcal{R}$; this leads to the following problem for Eq. (1.1):

$$\begin{cases} \mathcal{L}u - cu = \phi & \text{in } \Omega, \\ u = \gamma & \text{in } \mathcal{R}. \end{cases} \tag{1.3}$$

Similarly, for Eq. (1.2) we address the problem:

$$\begin{cases} \mathcal{L}u - cu - \partial_t u = f & \text{in } Q_T, \\ u = g & \text{in } \mathcal{R} \times (0, T], \\ u = u_0 & \text{in } [\Omega \cup \mathcal{R}] \times \{0\}. \end{cases} \tag{1.4}$$

Sufficient conditions for uniqueness or nonuniqueness of solutions to problems (1.3)–(1.4) have been given in [11,13,12]. These conditions are implicit in character, for they depend on the existence of suitable sub- and supersolutions to related elliptic problems, like the *first exit time problem*:

$$\begin{cases} \mathcal{L}U = -1 & \text{in } \Omega, \\ U = 0 & \text{on } \mathcal{R}. \end{cases} \tag{1.5}$$

In this paper we address the actual construction of such sub- and supersolutions, aiming to give explicit criteria for well-posedness of problems (1.3)–(1.4). Not surprisingly, the feasibility of this program depends on geometrical properties of the singular boundary $\mathcal{S}$ (in particular, on its dimension), as well as on the behavior of the coefficients of the operator $\mathcal{L}$ as the distance $d(x, \mathcal{S})$ goes to zero.

### *1.1. Assumptions*

Our assumptions concerning the set $\Omega$, the regular boundary $\mathcal{R}$ and the singular boundary $\mathcal{S}$ are summarized as follows:

$$(A_1) \quad \begin{cases} \text{(i)} \ \Omega \subseteq \mathbb{R}^n \text{ is open, bounded and connected;} \\ \text{(ii)} \ \partial\Omega = \mathcal{R} \cup \mathcal{S},\ \overline{\mathcal{R}} \cap \overline{\mathcal{S}} = \emptyset,\ \mathcal{S} \neq \emptyset; \\ \text{(iii)} \ \mathcal{R} \subseteq \partial\overline{\Omega},\ \Omega \text{ satisfies the outer sphere condition at } \mathcal{R}; \\ \text{(iv)} \ \mathcal{S} \text{ is a compact } k\text{-dimensional submanifold of } \mathbb{R}^n \text{ of class } C^3\ (k = 0, 1, \dots, n-1) \end{cases}$$

(we say that $\dim \mathcal{S} = 0$, if $\mathcal{S}$ is a finite union of points). Further assumptions will be needed below (see $(A_3)$).

Cases where different connected components of $\mathcal{S}$ are submanifolds of different dimension can also be considered; we omit the details.

It is natural to choose $\mathcal{R}$ as *the largest subset* of $\partial\Omega$ where ellipticity of the operator $\mathcal{L}$ holds (see assumptions $(E_1)$, $(E_2)$(ii) below); we do so in the following.

Denote by $C^{k,1}(B)$ the space of functions defined in a subset $B \subseteq \overline{\Omega}$, whose derivatives of order $\leqslant k$ $(k = 0, 1)$ are locally Lipschitz continuous in $B$. Concerning coefficients and data of the elliptic problem (1.3), we make the following assumptions:

$$(A_2)\quad \begin{cases} \text{(i)} & \rho \in C^{1,1}(\Omega \cup \mathcal{R}),\ \rho > 0 \ \text{ in } \ \Omega \cup \mathcal{R}; \\ \text{(ii)} & a_{ij} = a_{ji} \in C^{1,1}(\Omega \cup \mathcal{R}) \cap C^{0,1}(\overline{\Omega}),\ b_i \in C^{0,1}(\Omega \cup \mathcal{R}) \cap L^{\infty}(\Omega)\ (i, j = 1, \dots, n); \\ \text{(iii)} & c \in C(\Omega \cup \mathcal{R}),\ c \geqslant 0; \\ \text{(iv)} & \phi \in C(\Omega); \\ \text{(v)} & \gamma \in C(\mathcal{R}) \end{cases}$$

(the assumption $b_i \in L^{\infty}(\Omega)$ will be omitted in Theorem 2.12).

Our nonuniqueness results for problem (1.3) require ellipticity of the operator $\mathcal{L}$ in $\Omega$; therefore we assume:

$$(E_1)\quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0 \quad \text{for any } x \in \Omega \cup \mathcal{R}, \text{ and } (\xi_1, \dots, \xi_n) \neq 0.$$

On the other hand, the uniqueness results hold true even if ellipticity of $\mathcal{L}$ in $\Omega$ is lost. In this case we replace assumption $(E_1)$ by the following:

$$(E_2)\quad \begin{cases} \text{(i)} & \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant 0 \text{ for any } x \in \Omega \text{ and } (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \quad \sigma_{ij} \in C^1(\Omega) \\ & (i, j = 1, \dots, n); \\ \text{(ii)} & \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \mathcal{R} \text{ and } (\xi_1, \dots, \xi_n) \neq 0; \\ \text{(iii)} & \text{either } c > 0 \text{ in } \Omega \cup \mathcal{R}, \text{ or } c \geqslant 0 \text{ and } c + \sum_{i=1}^{n} \sigma_{ji}^2 > 0 \text{ in } \Omega \cup \mathcal{R} \text{ for some } j = 1, \dots, n; \end{cases}$$

here $\sigma \equiv (\sigma_{ij})$ denotes the square root of the matrix $A \equiv (a_{ij})$ (namely, $A(x) = \sigma(x)\sigma(x)^T$; $x \in \overline{\Omega}$). Assumption $(E_2)$ (in particular, $(E_2)$(iii)) enables us to use comparison results for *viscosity* sub- and supersolutions to second order degenerate elliptic equations, via an equivalence result proved in [6] (see [13]; for the parabolic case [12]).

### *1.2. Well-posedness conditions*

The following result was proved in [11] (see also [13]).

**Theorem 1.1.** *Let assumptions* $(A_1)$–$(A_2)$ *and* $(E_1)$ *be satisfied*; *suppose* $c \in L^{\infty}(\Omega)$. *Let there exists a supersolution* $V$ *of problem* (1.5) *such that*

$$\inf_{\Omega \cup \mathcal{R}} V = 0 < \inf_{\mathcal{R}} V. \tag{1.6}$$

*Then either no solutions, or infinitely many solutions of problem* (1.3) *exist.*

The proof of the above theorem shows that nonuniqueness depends on the possibility of prescribing the value of the solution of problem (1.3) at some point of the singular boundary $\mathcal{S}$. Typically, to have a well-posed problem boundary conditions must be prescribed on some subset $\mathcal{S}_1 \subset \mathcal{S}$, while on the complementary subset $\mathcal{S}_2$ the singular character of the operator does not allow to impose boundary data. Hence we make the following assumption:

$$(A_3)\quad \begin{cases} \text{(i)} & \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2,\ \overline{\mathcal{S}_1} \cap \overline{\mathcal{S}_2} = \emptyset; \\ \text{(ii)} & \mathcal{S}_j = \bigcup_{k=1}^{k_j} \mathcal{S}_j^k, \text{ where every } \mathcal{S}_j^k \text{ is connected and } \overline{\mathcal{S}_j^k} \cap \overline{\mathcal{S}_j^l} = \emptyset \text{ for any } k, l = 1, \dots, k_j, \\ & k \neq l, \text{ if } k_j \geqslant 2\ (j = 1, 2). \end{cases}$$

Then we consider the problem:

$$\begin{cases} \mathcal{L}u - cu = \phi & \text{in } \Omega, \\ u = \gamma & \text{in } \mathcal{R} \cup \mathcal{S}_1. \end{cases} \tag{1.7}$$

For the same reason we associate to the parabolic problem (1.4) the following:

$$\begin{cases} \mathcal{L}u - cu - \partial_t u = f & \text{in } Q_T, \\ u = g & \text{in } [\mathcal{R} \cup \mathcal{S}_1] \times (0, T], \\ u = u_0 & \text{in } [\Omega \cup \mathcal{R} \cup \mathcal{S}_1] \times \{0\}. \end{cases} \tag{1.8}$$

In the elliptic case, sufficient conditions for uniqueness of solutions to problem (1.7) have been proved in [13] (analogous results hold for the parabolic case, see [12]). Such conditions depend on the existence of subsolutions to the *homogeneous problem*:

$$\begin{cases} \mathcal{L}U = cU & \text{in } \Omega, \\ U = 0 & \text{on } \mathcal{R}, \end{cases} \tag{1.9}$$

and on their behavior as the distance $d(x, \mathcal{S}_2)$ goes to zero. Let us mention the following result.

**Theorem 1.2.** *Let assumptions* $(A_1)$*–*$(A_3)$*, and either* $(E_1)$ *or* $(E_2)$ *be satisfied. Suppose* $\mathcal{S}_2 \neq \emptyset$, $\gamma \in C(\mathcal{R} \cup \mathcal{S}_1)$*. Let there exist a subsolution* $Z \leqslant H < 0$ *of problem* (1.9)*. Then there exists at most one solution* $u$ *of problem* (1.7) *such that*

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{Z(x)} = 0. \tag{1.10}$$

Clearly, if $\mathcal{S}_1 = \emptyset$ we recover uniqueness conditions for problem (1.3). If $\mathcal{S}_1 \neq \emptyset$, existence of solutions to problem (1.7) implies nonuniqueness for problem (1.3), since $\overline{\mathcal{R}} \cap \overline{\mathcal{S}} = \emptyset$ by assumption and the boundary data on $\mathcal{S}_1$ can be arbitrarily chosen. This remark will be used below, e.g., in the proof of Theorems 2.20–2.21.

Let us mention for further purposes that problem (1.9) can be replaced by problem (1.5) in the above statement, if $c(x) \geqslant c_0 > 0$ in $\Omega$.

### *1.3. Outline of results*

As already pointed out, applying Theorems 1.1–1.2 to concrete cases calls for the actual construction of the sub- and supersolutions $V, Z$; this is the point we address below. Results analogous to Theorems 1.1–1.2 have been proved in [12] for the parabolic problems (1.4), (1.8), relying on the existence of the same functions $V, Z$ as above. Therefore our criteria for well-posedness are conceptually the same both for the elliptic and the parabolic case (see Sections 2–3).

Our main results for the elliptic case can be described as follows (for the parabolic case, see Section 3).

(a) If $n \geqslant 2$, $\dim \mathcal{S} \leqslant n - 2$ and the *orthogonal rank* of the diffusion matrix $A$ is at least 2 on $\mathcal{S}$, there exists at most one bounded solution of problem (1.3) (actually, the uniqueness class is larger; see Definition 2.11 and Theorem 2.12). This result extends Theorem 4.1, Chapter 11 in [4], which was proved under more restrictive assumptions by stochastic methods; it also extends the results in [7], where $A$ was uniformly elliptic.

We stress that the above uniqueness result holds *without imposing any additional condition at* $\mathcal{S}$. In the parlance of [4], conditions at $\mathcal{S}$ are unnecessary for uniqueness since $\mathcal{S}$ is "too thin", hence *non-attainable* by trajectories of the Markov process generated by the operator $\mathcal{L}$.

(b) If $\dim \mathcal{S} = n - 1$, well-posedness crucially depends on the behavior of the coefficients of $\mathcal{L}$ near $\mathcal{S}$. Roughly speaking, if "diffusion near $\mathcal{S}$ is low" (see Theorem 2.16, in particular condition (2.16)–(2.17)), no additional conditions at $\mathcal{S}$ are needed to ensure uniqueness of problem (1.3), much as in the case $n \geqslant 2$, $\dim \mathcal{S} \leqslant n - 2$. The opposite holds when "diffusion near $\mathcal{S}$ is high": in this case boundary conditions on some part of $\mathcal{S}$ are necessary to make the problem well posed (see Theorem 2.18 and condition (2.20)–(2.21)). An interesting model case is when $\mathcal{L} = \frac{1}{\rho}\Delta$: if $\rho(x) \sim [d(x, \mathcal{S})]^{-\alpha}$ for some $\alpha \geqslant 2$, Theorem 2.16 applies and no additional conditions at $\mathcal{S}$ are needed; the opposite holds, if $\rho(x) \sim [d(x, \mathcal{S})]^{-\alpha}$ with $\alpha < 2$, so that Theorem 2.18 applies (see Example (c) in Section 6).

In the light of the previous results, $\dim \mathcal{S} = n - 2$ is critical for well-posedness of problem (1.3) in the class of bounded solutions. This is not surprising, since $(n - 2)$ is the critical dimension for studying sets of zero capacity with respect to uniformly elliptic second order operators (see [15]). In such case the role of capacity to study uniqueness of the bounded Cauchy problem is well understood (e.g., see [5]). We are not aware of similar results in the present more general case (however, see Remark 2.15 below for the particular case $\mathcal{L} = \frac{1}{\rho}\Delta$).

Also the role of the behavior of $\rho$ near the singular boundary when $\dim \mathcal{S} = n - 1$ is not unexpected. In fact, the above condition $\rho(x) \sim [d(x, \mathcal{S})]^{-\alpha}$ $(\alpha < 2)$ was considered in [14], where the generation of semigroups in $L^\infty(\Omega)$ by second order operators, with coefficients possibly vanishing at $\partial\Omega$, was investigated.

Let us mention that results analogous to Theorems 1.1–1.2 also hold for unbounded domains (see [13]). Accordingly, several results we state below can be extended to domains of this kind; we leave their formulation to the reader.

The paper is organized as follows. In Section 2 first we introduce some definitions and related results existing in the literature, then we state our main results concerning elliptic problems. The same is done for parabolic problems in Section 3. Proofs are given in Sections 4 and 5, depending on the assumption made on the dimension of the singular manifold. Finally, a few examples are discussed in Section 6.

## 2. Elliptic problems

### *2.1. Mathematical framework and auxiliary results*

#### *2.1.1. Sub- and supersolutions*

Let us make precise the definition of solution to the above mentioned elliptic problems. Denote by $\mathcal{M}^*$ the formal adjoint of the operator $\mathcal{M}$, namely:

$$\mathcal{M}^* u \equiv \sum_{i,j=1}^{n} \frac{\partial^2(a_{ij}u)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial(b_i u)}{\partial x_i}.$$

**Definition 2.1.** By a *subsolution* to Eq. (1.1) we mean any function $u \in C(\Omega)$ such that

$$\int_\Omega u\{\mathcal{M}^*\psi - \rho c\psi\}\,dx \geqslant \int_\Omega \rho\phi\psi\,dx \tag{2.1}$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$. *Supersolutions* of (1.1) are defined replacing "$\geqslant$" by "$\leqslant$" in (2.1). A function $u$ is a *solution* of (1.1) if it is both a sub- and a supersolution.

**Definition 2.2.** Let $\mathcal{R} \subseteq \mathcal{E} \subseteq \partial\Omega$, $\gamma \in C(\mathcal{E})$. By a *subsolution* to the problem

$$\begin{cases} \mathcal{L}u - cu = \phi & \text{in } \Omega, \\ u = \gamma & \text{on } \mathcal{E} \end{cases} \tag{2.2}$$

we mean any function $u \in C(\Omega \cup \mathcal{E})$ such that

(i) $u$ is a subsolution of Eq. (1.1);
(ii) $u \leqslant \gamma$ on $\mathcal{E}$.

*Supersolutions* and *solutions* of (2.2) are similarly defined.

#### *2.1.2. Attracting boundaries and barriers*

Let $\Sigma \subseteq \partial\Omega$; define:

$$\Sigma^\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \Sigma) < \varepsilon\} \quad (\varepsilon > 0).$$

Also set

$$B_r(x^0) := \{x \in \mathbb{R}^n \mid |x - x^0| < r\} \quad (x^0 \in \mathbb{R}^n),$$
$$\mathcal{B}_r(y^0) := B_r(y^0) \cap \mathcal{S} \quad (y^0 \in \mathcal{S}).$$

Let us introduce, for use in the sequel, the following definitions (see [8]).

**Definition 2.3.** We say that $\Sigma \subseteq \partial\overline{\Omega}$ is *attracting*, if there exist $\varepsilon > 0$ and a supersolution $V \in C(\overline{\Sigma^\varepsilon})$ of the equation

$$\mathcal{L}u - cu = -1 \quad \text{in } \Sigma^\varepsilon, \tag{2.3}$$

such that

$$V > 0 \quad \text{in } \overline{\Sigma^\varepsilon} \setminus \Sigma, \qquad V = 0 \quad \text{on } \Sigma.$$

**Definition 2.4.** Let $x^0 \in \partial\overline{\Omega}$. A function $h \in C(\overline{\Omega \cap B_r(x^0)})$ is called a *barrier* at $x^0$ if:

(i) $h$ is a supersolution of

$$\mathcal{L}u - cu = -1 \quad \text{in } \Omega \cap B_r\left(x^0\right),$$

(ii) there holds

$$h > 0 \quad \text{in } \overline{\Omega \cap B_r(x^0)} \setminus \left\{x^0\right\}, \qquad h\left(x^0\right) = 0.$$

If $\Sigma$ is attracting, the function $V$ can be viewed as a barrier for the whole of $\Sigma$.

Let us state the following result, concerning existence of solutions to problem (1.7) (or (1.3), if $\mathcal{S}_1 = \emptyset$; see [13]).

**Theorem 2.5.** *Let $\mathcal{S}_1 \subseteq \partial\overline{\Omega}$ be attracting and assumptions $(A_1)$–$(A_3)$, $(E_1)$ be satisfied. In addition, suppose*:

(a) $c \in L^\infty(\mathcal{S}_1^\varepsilon)$ *for some* $\varepsilon > 0$;
(b) $\phi \in L^\infty(\Omega)$;
(c) $\gamma \in C(\mathcal{R} \cup \mathcal{S}_1)$.

*Let there exist a positive supersolution $F \in C(\Omega \cup \mathcal{R}) \cap L^\infty(\mathcal{S}_1^\varepsilon)$ of the equation*:

$$\mathcal{L}u - cu = -1 \quad \textit{in } \Omega. \tag{2.4}$$

*Then there exists a solution of problem* (1.7)*, provided that*

$$\gamma = \textit{constant} \quad \textit{on } \mathcal{S}_1. \tag{2.5}$$

*Condition* (2.5) *is unnecessary, if a barrier exists at any point of* $\mathcal{S}_1$.

**Remark 2.6.** If $c(x) \geqslant c_0 > 0$ in $\Omega$, we can take $F \equiv \frac{1}{c_0}$ in $\overline{\Omega}$ as a supersolution of (2.4).

Under the assumptions of Theorem 2.5, since $\mathcal{S}_1$ is attracting, *constant* Dirichlet data can be prescribed on it. Moreover, if a barrier exists at any point of $\mathcal{S}_1$ also general Dirichlet data can be prescribed, but this need not be the case without this additional requirement (e.g., see [13] for an example). If the coefficients $a_{ij}, b_i$ are bounded and $\rho$ is bounded away from zero in $\mathcal{S}_1^\varepsilon$ for some $\varepsilon > 0$, a barrier exists at any point of $\mathcal{S}_1$ (see [8,13]; see also the proof of Proposition 2.7 below).

*2.1.3. Revisiting classical results*

The above remarks are deeply connected with the approach developed in [2] to investigate uniqueness for problem (2.2). As in [2], let the following assumptions be satisfied:

$$(F_1)\quad \begin{cases} \text{(i)}\ \ \Omega \text{ open, bounded and connected, } \partial\Omega = \partial\overline{\Omega}; \\ \text{(ii)}\ \ \partial\Omega = \bigcup_{h=1}^m \Gamma_h;\ \text{each } \Gamma_h \text{ is a regular } (n-1)\text{-dimensional submanifold with boundary} \\ \qquad \partial\Gamma_h\ (h = 1, 2, \ldots, m); \\ \text{(iii)}\ \ \Gamma_h \cap \Gamma_k = \partial\Gamma_h \cap \partial\Gamma_k \text{ for any } h, k = 1, \ldots, m,\ h \neq k, \end{cases}$$

and

$$(F_2)\quad \begin{cases} \text{(i)}\ \ \rho \in C^2(\overline{\Omega}),\ \rho > 0 \text{ in } \overline{\Omega}; \\ \text{(ii)}\ \ a_{ij} = a_{ji} \in C^2(\overline{\Omega}),\ b_i \in C^1(\overline{\Omega}); \\ \text{(iii)}\ \ c \in C(\overline{\Omega}),\ c \geqslant 0, \end{cases}$$

$$(F_3)\quad \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqslant 0 \quad \text{for any } x \in \Omega \text{ and } (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.$$

Define for any $x \in \Gamma_h \setminus \partial\Gamma_h$ $(h = 1, \ldots, m)$:

$$\alpha_F(x) := \sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x)\nu_j(x),$$
$$\beta_F(x) := \sum_{i=1}^{n}\left[b_i(x) - \sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j}\right]\nu_i(x),$$

where $\nu(x) \equiv (\nu_1(x), \dots, \nu_n(x))$ denotes the outer normal to $\Omega$ at $x \in \Gamma_h \setminus \partial\Gamma_h$; then extend the definition of $\alpha_F$, $\beta_F$ to $\partial\Gamma_h$ by continuity. Observe that the extensions of $\alpha_F$, $\beta_F$ to $\partial\Gamma_h$ for different values of $h = 1, \dots, m$, need not agree on the intersection of the boundaries.

Set

$$\Sigma_1 := \{x \in \partial\Omega \mid \alpha_F(x) = 0,\ \beta_F(x) \leqslant 0\}, \tag{2.6}$$
$$\Sigma_2 := \{x \in \partial\Omega \mid \alpha_F(x) = 0,\ \beta_F(x) > 0\}, \tag{2.7}$$
$$\Sigma_3 := \{x \in \partial\Omega \mid \alpha_F(x) > 0\} = \partial\Omega \setminus [\Sigma_1 \cup \Sigma_2]. \tag{2.8}$$

Observe that $\Sigma_3$ contains the regular boundary $\mathcal{R}$, if $(F_1)$ holds. Moreover, the *drift trajectories* (e.g., see [1]) do not point outwards at the points of $\Sigma_1$, but they do at those of $\Sigma_2$.

The following result will be proved (see Section 5).

**Proposition 2.7.** *Let assumptions* $(F_1)$–$(F_3)$ *be satisfied*; *let* $\sigma_{ij} \in C^1(\overline{\Sigma_2{}^{\varepsilon}})$ *for some* $\varepsilon > 0$ $(i, j = 1, \dots, n)$. *Let* $\Sigma$ *be a smooth connected component of* $\partial\Omega$, *such that* $\Sigma \subseteq \Sigma_2$. *Then*:

(i) $\Sigma$ *is attracting*;
(ii) *for any* $x^0 \in \Sigma$ *there exists a barrier.*

The proof of claim (i) relies on the fact that some multiple of the distance $d(\cdot, \Sigma)$ is a supersolution of Eq. (2.3) (see Definition 2.3); claim (ii) follows by a standard argument from the attractivity of $\Sigma$ and the boundedness of the coefficients $\rho$, $a_{ij}$, $b_i$.

The proof of the following result is similar to that of Proposition 2.7, thus it will be omitted. A related result can be found in Lemma 2.7.1 of [10].

**Proposition 2.8.** *Let assumptions* $(F_1)$–$(F_3)$ *be satisfied. Let* $\Sigma$ *be a smooth connected component of* $\partial\Omega$, *such that* $\Sigma \subseteq \Sigma_3$. *Then for any* $x^0 \in \Sigma$ *there exists a barrier.*

If $\Sigma \subseteq \Sigma_1$, the distance $d(\cdot, \Sigma)$ can be used to construct a subsolution of problem (1.9). This is the content of the following proposition, where $\chi \in C^2(\overline{\Omega})$, $0 \leqslant \chi \leqslant 1$ is any function such that

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \Sigma^{\varepsilon/2}, \\ 0 & \text{if } x \in \Omega \setminus \Sigma^{\varepsilon}\ (\varepsilon > 0). \end{cases} \tag{2.9}$$

Similar results can be found in Theorem 2.7.1 of [10] and in Chapter 9, vol. I of [4].

**Proposition 2.9.** *Let assumptions* $(F_1)$–$(F_3)$ *be satisfied*; *for some* $\varepsilon_0 > 0$ *let* $\sigma_{ij} \in C^1(\overline{\Sigma_1{}^{\varepsilon_0}})$ $(i, j = 1, \dots, n)$. *Let* $\Sigma$ *be a smooth connected component of* $\partial\Omega$, *such that* $\Sigma \subseteq \Sigma_1$; *suppose* $c(x) \geqslant c_0 > 0$ *for any* $x \in \Omega$. *Then for any* $\alpha > 0$ *sufficiently small and* $H > 0$ *large enough there exists* $\varepsilon \in (0, \varepsilon_0)$ *such that the function,*

$$Z(x) := -[d(x, \Sigma)]^{-\alpha}\chi(x) - H \quad (x \in \overline{\Omega} \setminus \Sigma) \tag{2.10}$$

(*where* $\chi = \chi_\varepsilon$ *satisfies* (2.9)) *is a subsolution of problem* (1.9).

In view of Theorem 1.2, if $\overline{\mathcal{S}_2} \subseteq \Sigma_1$ and the assumptions of Proposition 2.9 are satisfied, we expect uniqueness of solutions to problem (1.7) such that

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{[d(x, \mathcal{S}_2)]^{-\alpha}} = 0 \tag{2.11}$$

(with $\alpha > 0$ sufficiently small), thus in particular uniqueness of bounded solutions. In fact, this is the content of Theorem 2.23 below. Observe that Proposition 2.9 is in agreement with the following uniqueness result, which was proved in [2].

**Theorem 2.10.** *Let assumptions $(F_1)$–$(F_3)$ be satisfied and $c > 0$ in $\overline{\Omega}$. Suppose that the Gauss–Green identity applies in $\overline{\Omega}$. Then problem* (2.2) *with $\mathcal{E} = \Sigma_2 \cup \Sigma_3$ admits at most one solution in the space $C_{\mathcal{L}} := \{u \in C^1(\overline{\Omega}) \cap C^2(\Omega) \mid \mathcal{L}u \in L^\infty(\Omega)\}$.*

### 2.2. Main results: Singular manifolds of low dimension

To state our results we need some preliminary remarks. Set $k \equiv \dim \mathcal{S}$; denote by $\mathcal{M}_m$ the linear space of $m \times m$ matrices with real entries ($m \in \mathbb{N}$). For any fixed $y \in \mathcal{S}$ there exist orthonormal vectors $\eta^{(1)}(y), \ldots, \eta^{(n-k)}(y) \in \mathbb{R}^n$, which are orthogonal to $\mathcal{S}$ at $y$. Consider the matrix $A_\perp(y) \equiv (\alpha_{lm}(y)) \in \mathcal{M}_{n-k}$, where

$$\alpha_{lm}(y) := \sum_{i,j=1}^{n} a_{ij}(y)\eta_i^{(l)}(y)\eta_j^{(m)}(y) \quad (l, m = 1, \ldots, n-k;\ y \in \mathcal{S}).$$

Let us make the following definition (see [4]).

**Definition 2.11.** Let $y \in \mathcal{S}$. The rank $r(y)$ of the matrix $A_\perp(y)$ is called the *orthogonal rank of the diffusion matrix $A$ at $y$*.

The above definition is well-posed, for $r(y)$ is independent of the choice of the set $\{\eta^{(l)}(y) \mid l = 1, \ldots, n-k\}$; observe that $r(y) \leqslant n-k$. In view of assumption $(A_1)$(iv), there exist $y^1, \ldots, y^N \in \mathcal{S}$ such that

$$\begin{cases} \mathcal{S} \text{ is the union of the graphs } \mathcal{U}_i \text{ of } C^3 \text{ functions, say} \\ \phi^{(i)} : B_{R_i}(y_1^i, \ldots, y_k^i) \subset \mathbb{R}^k \to \mathbb{R}^{n-k}, \quad \phi^{(i)} \equiv (\phi_{k+1}^{(i)}, \ldots, \phi_n^{(i)}) \\ \quad (i = 1, \ldots, N), \text{ up to reorderings of the coordinates.} \end{cases} \tag{2.12}$$

We shall use the following assumption:

$$(A_4) \quad \begin{cases} \text{(i)} \ n \geqslant 2, \dim \mathcal{S} \leqslant n-2; \\ \text{(ii)} \ r(y) \geqslant 2 \text{ for any } y \in \mathcal{S}; \\ \text{(iii)} \ \text{for any } y \in \mathcal{U}_i \ (i = 1, \ldots, N) \text{ there exist orthonormal vectors } \eta^{(1)}(y), \ldots, \eta^{(n-k)}(y) \in \mathbb{R}^n, \\ \qquad \text{which are orthogonal to } \mathcal{S} \text{ at } y,\ \eta^{(l)}(\cdot) \in C^2(\mathcal{U}_i; \mathbb{R}^n)\ (l = 1, \ldots, n-k), \text{ such that the matrix} \\ \qquad A_\perp(\cdot) \text{ has unit eigenvectors of class } C^2(\mathcal{U}_i; \mathbb{R}^{n-k}), \end{cases}$$

(here the notation in (2.12) has been used).

Now we can state the following:

**Theorem 2.12.** *Let assumptions $(A_1)$–$(A_4)$ be satisfied, with $(A_2)$(ii) replaced by:*

$$(A_2)(\mathrm{ii})' \quad \begin{cases} a_{ij} = a_{ji} \in C^2(\overline{\Omega}), \quad b_i \in C^{0,1}(\Omega \cup \mathcal{R}); \\ \text{there exist } B_0 > 0 \text{ and } \beta \in [0,1) \text{ such that} \\ |b_i(x)| \leqslant \frac{B_0}{[d(x,\mathcal{S})]^\beta} \quad \text{for any } x \in \Omega\ (i, j = 1, \ldots, n). \end{cases}$$

*Moreover, let either $(E_1)$ or $(E_2)$ hold; if $(E_2)$ holds, let $c(x) > 0$ for any $x \in \Omega$. Then,*

(i) *there exists at most one solution $u$ of problem* (1.3) *such that*

$$\lim_{d(x,\mathcal{S}) \to 0} \frac{u(x)}{\log[d(x,\mathcal{S})]} = 0; \tag{2.13}$$

(ii) *if $\alpha := \inf_{y \in \mathcal{S}} r(y) - 2 \geqslant 1$, there exists at most one solution $u$ of problem* (1.3) *such that*

$$\lim_{d(x,\mathcal{S}) \to 0} \frac{u(x)}{[d(x,\mathcal{S})]^{-\alpha}} = 0. \tag{2.14}$$

*In particular, problem* (1.3) *has at most one bounded solution.*

A simple application of Theorem 2.12 is given in Section 6, Example (a). Example (b) in the same section shows that the limit value $\beta = 1$ in assumption $(A_2)$(ii)$'$ above is not allowed.

Theorem 2.12 follows from Theorem 1.2 (with $\mathcal{S}_2 = \mathcal{S}$), if we exhibit a subsolution $Z$ of problem (1.9) diverging like $\log[d(x,\mathcal{S})]$, or like $[d(x,\mathcal{S})]^{-\alpha}$ if $\alpha = \inf_{y\in\mathcal{S}} r(y) - 2 \geqslant 1$, as $d(x,\mathcal{S}) \to 0$; this is done in Section 4. Remarkably, the construction of $Z$ does not require any assumption on the behavior of $\rho$ near $\mathcal{S}$ (which instead plays a role when $\dim \mathcal{S} = n-1$; see Section 2.3).

Let us also mention the following well-posedness result, which follows immediately from Theorems 2.12, 2.5 (with $\mathcal{S}_1 = \emptyset$) and Remark 2.6.

**Theorem 2.13.** *Let assumptions* $(A_1)$–$(A_4)$, *with* $(A_2)$(ii) *replaced by* $(A_2)$(ii)$'$, *and* $(E_1)$ *be satisfied*; *suppose* $\phi \in L^\infty(\Omega)$ *and* $c(x) \geqslant c_0 > 0$ *for any* $x \in \Omega$. *Then there exists a unique bounded solution of problem* (1.3).

As an example, it is informative to discuss the above situation when $A = (\delta_{ij})$. In this case problem (1.3) reads:

$$\begin{cases} \dfrac{1}{\rho}\Delta u - cu = \phi & \text{in } \Omega, \\ u = \gamma & \text{in } \mathcal{R}. \end{cases} \tag{2.15}$$

Since $r(y) = n-k$ $(y \in \mathcal{S})$, assumption $(A_4)$ reduces to $(A_4)$(i); then we have the following refinement of Theorem 2.12.

**Corollary 2.14.** *Let assumptions* $(A_1)$–$(A_3)$ *be satisfied, with* $(A_2)$(ii) *replaced by* $(A_2)$(ii)$'$; *suppose* $\dim \mathcal{S} \leqslant n-2$. *Then,*

(i) *there exists at most one solution u of problem* (2.15) *satisfying* (2.13);
(ii) *if* $\dim \mathcal{S} \equiv k \leqslant n-3$, *there exists at most one solution u of problem* (2.15) *such that*

$$\lim_{d(x,\mathcal{S})\to 0} \frac{u(x)}{[d(x,\mathcal{S})]^{2-(n-k)}} = 0.$$

*In particular, there exists at most one bounded solution of problem* (2.15).

**Remark 2.15.** For $n \geqslant 2$, if $\dim \mathcal{S} < n-2$, or if $\dim \mathcal{S} = n-2$ and the Hausdorff measure $\mathcal{H}^{N-2}(\mathcal{S})$ is finite, the capacity $\operatorname{cap}_\Delta \mathcal{S}$ is zero and it is well known that

$$\operatorname{cap}_\Delta \mathcal{S} = 0 \quad \Leftrightarrow \quad \begin{cases} \text{there exists } u \in C^2(\Omega \cup \mathcal{R}) \text{ such that } u > 0 \text{ in } \Omega \cup \mathcal{R}, \\ \Delta u \leqslant 0 \text{ in } \Omega \cup \mathcal{R},\ u(x) \to +\infty \text{ as } d(x,\mathcal{S}) \to 0. \end{cases}$$

Hence $Z := -u - 1$ is a subsolution of problem (1.9) which diverges as $d(x,\mathcal{S}) \to 0$, and by Theorem 1.2 there exists at most one bounded solution of problem (2.15), in agreement with the above corollary. Similar remarks hold for uniformly elliptic operators with sufficiently smooth coefficients (e.g., see [15] and references therein).

### *2.3. Main results: Singular manifolds of high dimension*

Let us now address the case $\dim \mathcal{S} = n-1$. We shall prove the following uniqueness result:

**Theorem 2.16.** *Let* $\dim \mathcal{S} = n-1$; *let assumptions* $(A_1)$–$(A_3)$, *and either* $(E_1)$ *or* $(E_2)$ *be satisfied. Assume* $\mathcal{S}_2 \neq \emptyset$. *In addition, suppose the following*:

(a) *there exist* $\bar{\varepsilon} > 0$ *and a positive, continuous function* $\underline{\rho}$ *satisfying*

$$\int_0^{\bar{\varepsilon}} \eta \underline{\rho}(\eta)\, d\eta = +\infty \tag{2.16}$$

*such that*

$$\rho(x) \geqslant \underline{\rho}\big(d(x, \mathcal{S}_2)\big) \quad \textit{for any } x \in \mathcal{S}_2^{\bar{\varepsilon}}; \tag{2.17}$$

(b) $c(x) \geqslant c_0 > 0$ *for any* $x \in \Omega$;
(c) $\gamma \in C(\mathcal{R} \cup \mathcal{S}_1)$.

*Then,*

(i) *the function*

$$P(\zeta) := \int_{\zeta}^{\bar{\varepsilon}} (\eta - \zeta)\underline{\rho}(\eta)\, d\eta \quad \big(\zeta \in (0, \bar{\varepsilon})\big) \tag{2.18}$$

*diverges as* $\zeta \to 0^+$;
(ii) *there exists at most one solution* $u$ *of problem* (1.7) *such that*

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{P(d(x, \mathcal{S}_2))} = 0. \tag{2.19}$$

*In particular, problem* (1.7) *has at most one bounded solution.*

In view of Theorem 1.2, Theorem 2.16 follows by constructing a suitable subsolution $Z$ of problem (1.9) such that $|Z(x)|$ diverges with the same order of $P(d(x, \mathcal{S}_2))$ as $d(x, \mathcal{S}_2) \to 0$ (see Section 5).

**Remark 2.17.** A natural choice in Theorem 2.16 is $\underline{\rho}(\eta) = \eta^{-\sigma}$, $\sigma \geqslant 2$. Then,

(i) if $\sigma = 2$, there exists at most one solution $u$ of problem (1.7) such that

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{\log[d(x, \mathcal{S}_2)]} = 0;$$

(ii) if $\sigma > 2$, there exists at most one solution $u$ of problem (1.7) such that

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{u(x)}{[d(x, \mathcal{S}_2)]^{2-\sigma}} = 0.$$

In such cases the order of divergence of $P(\zeta)$ as $\zeta \to 0^+$ is the same as $Q(\zeta) := \int_{\zeta}^{\bar{\varepsilon}} \eta\underline{\rho}(\eta)\, d\eta$, although in general it can be lower (e.g., take $\underline{\rho}(\eta) := e^{1/\eta}/\eta^3$).

In view of Theorem 2.16, no additional conditions at $\mathcal{S}_2$ are needed to ensure uniqueness of bounded solutions to problem (1.7), if (2.16)–(2.17) hold. If $\mathcal{S}_1 = \emptyset$, the situation is qualitatively the same as for $\dim \mathcal{S} \leqslant n - 2$ (see Theorem 2.12).

It is natural to investigate the complementary situation—namely, when conditions (2.20)–(2.21) below are satisfied. As it can be expected, boundary conditions at $\mathcal{S}_1$ are necessary in this case to have a well posed problem. In other words, nonuniqueness holds for problem (1.3), which lacks such conditions.

We address this situation strengthening assumption $(E_1)$, i.e., requiring it in $\overline{\Omega}$ and not only in $\Omega \cup \mathcal{R}$. This is equivalent to assume that there exists $\alpha > 0$ such that

$$(E_3) \qquad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \alpha|\xi|^2 \quad \text{for any } x \in \overline{\Omega} \text{ and } (\xi_1, \dots, \xi_n) \neq 0.$$

Then we have the following:

**Theorem 2.18.** *Let* $\dim \mathcal{S} = n - 1$. *Let assumptions* $(A_1)$–$(A_3)$ *and* $(E_3)$ *be satisfied*; *assume* $\mathcal{S}_2 = \emptyset$. *In addition, suppose the following*:

(a) *there exist* $\bar{\varepsilon} > 0$ *and a positive continuous function* $\bar{\rho}$ *satisfying*

$$\int_0^{\bar{\varepsilon}} \eta \bar{\rho}(\eta)\, d\eta < +\infty \tag{2.20}$$

*such that*

$$\rho(x) \leqslant \bar{\rho}\big(d(x, \mathcal{S}_1)\big) \quad \textit{for any } x \in \mathcal{S}_1^{\bar{\varepsilon}}; \tag{2.21}$$

(b) $c \in L^\infty(\Omega)$.

*Then either no solutions, or infinitely many solutions of problem* (1.3) *exist.*

**Remark 2.19.** A natural choice in Theorem 2.18 is $\bar{\rho}(\eta) = \eta^{-\sigma}$, $\sigma < 2$.

Theorem 2.18 is proved by constructing a positive supersolution $V$ of problem (1.5) with the properties assumed in Theorem 1.1. Clearly, $V$ is also a supersolution of Eq. (2.4)—namely, we can take $F = V$ in Theorem 2.5, which gives (with $\mathcal{S}_1 = \emptyset$) sufficient conditions for the existence of solutions to problem (1.3). Then from Theorem 2.18 we immediately obtain the following nonuniqueness result.

**Theorem 2.20.** *Let the assumptions of Theorem* 2.18 *be satisfied. In addition, suppose* $\phi \in L^\infty(\Omega)$. *Then infinitely many solutions of problem* (1.3) *exist.*

Indeed in the proof of Theorem 2.18 we construct a supersolution $V$ which shows that $\mathcal{S}_1 \subseteq \partial\overline{\Omega}$ is attracting. This suggests a different approach, i.e., assuming $\mathcal{S}_1 \subseteq \Sigma_2$ and using Proposition 2.7 (with $\Sigma$ a connected component of $\mathcal{S}_1$) to prove its attractivity, instead of assuming $(E_3)$ and "high diffusion near $\mathcal{S}_1$" as in (2.20)–(2.21). In this way we obtain the following nonuniqueness result.

**Theorem 2.21.** *Let* $\dim \mathcal{S} = n-1$, $\mathcal{S}_1 \neq \emptyset$ *and* $\mathcal{S}_1 \subseteq \Sigma_2$. *Let assumptions* $(A_1)$, $(E_1)$, $(F_1)$–$(F_2)$ *be satisfied and* $\sigma_{ij} \in C^1(\overline{\mathcal{S}_1^{\varepsilon}})$ *for some* $\varepsilon > 0$ $(i, j = 1, \ldots, n)$. *In addition, suppose*:

(a) $c(x) \geqslant c_0 > 0$ *for any* $x \in \Omega$;
(b) $\phi \in L^\infty(\Omega)$;
(c) $\gamma \in C(\mathcal{R})$.

*Then infinitely many solutions of problem* (1.3) *exist.*

**Remark 2.22.** The assumptions of Proposition 2.7 and hence of Theorem 2.21 can be weakened assuming $(A_2)$–$(A_3)$ instead of $(F_1)$–$(F_2)$, and supposing in addition:

(a) $a_{ij} \in C^{1,1}(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$, $b_i \in C^{0,1}(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$, $\sigma_{ij} \in C^1(\overline{\mathcal{S}_1^{\varepsilon}})$ for some $\varepsilon > 0$;
(b) $0 < \rho_0 \leqslant \rho(x) \leqslant \rho_1 < +\infty$ for any $x \in \mathcal{S}_1^{\varepsilon}$;
(c) $\rho \in C^{1,1}(\overline{\mathcal{S}_1^{\varepsilon}})$, $c \in L^\infty(\mathcal{S}_1^{\varepsilon})$ for some $\varepsilon > 0$.

Theorem 2.21 establishes nonuniqueness for problem (1.3), if boundary data are not prescribed on points of $\mathcal{S}$ where drift trajectories point outwards. On the other hand, there is uniqueness of *bounded* solutions to (1.3), if drift trajectories do not point outwards at any point of $\mathcal{S}$; this is a particular consequence of the following theorem, which relies on Proposition 2.9 and Remark 2.6.

**Theorem 2.23.** *Let* $\dim \mathcal{S} = n-1$, $\mathcal{S}_2 \neq \emptyset$ *and* $\mathcal{S}_2 \subseteq \Sigma_1$. *Let assumptions* $(A_1)$, $(E_2)$, $(F_1)$–$(F_2)$ *be satisfied and* $\sigma_{ij} \in C^1(\overline{\mathcal{S}_2^{\varepsilon}})$ *for some* $\varepsilon > 0$ $(i, j = 1, \ldots, n)$; *moreover, suppose* $c(x) \geqslant c_0 > 0$ *for any* $x \in \Omega$, $\gamma \in C(\mathcal{R} \cup \mathcal{S}_1)$. *Then there exists at most one solution of problem* (1.7) *satisfying condition* (2.11) ($\alpha > 0$ *sufficiently small*). *In particular, there exists at most one bounded solution of problem* (1.7).

A comparison between the results of Theorems 2.16, 2.20 and those of Theorems 2.21, 2.23 is given in Section 6, Example (c).

## 3. Parabolic problems

Results analogous to those above hold for parabolic problems (1.4), (1.8); the present section is devoted to the statement of the main of them.

We always assume the coefficients of the operator $\mathcal{L}$ to be independent of time. Concerning coefficients and data of the problems, let us state the counterpart of assumption $(A_2)$ for the parabolic problem, namely:

$$(A_5)\quad \begin{cases} \text{(i)}\ \ \rho \in C^{1,1}(\Omega \cup \mathcal{R}),\ \rho > 0 \text{ in } \ \Omega \cup \mathcal{R}; \\ \text{(ii)}\ \ a_{ij} = a_{ji} \in C^{1,1}(\Omega \cup \mathcal{R}) \cap C^{0,1}(\overline{\Omega}),\ b_i \in C^{0,1}(\Omega \cup \mathcal{R}) \cap L^\infty(\Omega)\ (i,j = 1,\dots,n); \\ \text{(iii)}\ \ c \in C(\Omega \cup \mathcal{R}),\ c \geqslant c_1 > -\infty; \\ \text{(iv)}\ \ f \in C(\overline{\Omega} \times [0,T]); \\ \text{(v)}\ \ g \in C(\mathcal{R} \times [0,T]),\ u_0 \in C(\Omega \cup \mathcal{R}); \\ \text{(vi)}\ \ g(x,0) = u_0(x) \text{ for any } x \in \mathcal{R}, \end{cases}$$

where (i)–(iii) coincide with those in $(A_2)$, apart from the sign condition on $c$ which is not needed anymore. Concerning ellipticity, we require the following weaker assumption (which coincides with $(E_2)$(i), (ii)):

$$(E_4)\quad \begin{cases} \text{(i)}\ \ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqslant 0 \text{ for any } x \in \Omega \text{ and } (\xi_1,\dots,\xi_n) \in \mathbb{R}^n, \quad \sigma_{ij} \in C^1(\Omega) \\ \qquad (i,j = 1,\dots,n); \\ \text{(ii)}\ \ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j > 0 \text{ for any } x \in \mathcal{R} \text{ and } (\xi_1,\dots,\xi_n) \neq 0. \end{cases}$$

Let us make the following definitions:

**Definition 3.1.** By a *subsolution* to Eq. (1.2) we mean any function $u \in C(\Omega \times (0,T])$ such that

$$\int\limits_{\Omega \times (0,T)} u\{\mathcal{M}^*\psi - \rho c\psi + \rho\partial_t\psi\}\,dx\,dt \geqslant \int\limits_{\Omega \times (0,T)} \rho f\psi\,dx\,dt, \tag{3.1}$$

for any $\psi \in C_0^\infty(\Omega \times (0,T))$, $\psi \geqslant 0$. *Supersolutions* of (1.2) are defined replacing "$\geqslant$" by "$\leqslant$" in (3.1). A function $u$ is a *solution* of (1.2) if it is both a sub- and a supersolution.

**Definition 3.2.** Let $\mathcal{R} \subseteq \mathcal{E} \subseteq \partial\Omega$, $g \in C(\mathcal{E} \times [0,T])$, $u_0 \in C(\Omega \cup \mathcal{E})$, $g(x,0) = u_0(x) (x \in \mathcal{E})$. By a *subsolution* to the problem,

$$\begin{cases} \mathcal{L}u - cu - \partial_t u = f & \text{in } \Omega \times (0,T), \\ u = g & \text{in } \mathcal{E} \times (0,T], \\ u = u_0 & \text{in } (\Omega \cup \mathcal{E}) \times \{0\}, \end{cases} \tag{3.2}$$

we mean any function $u \in C((\Omega \cup \mathcal{E}) \times [0,T])$ such that

(i) $u$ is a subsolution of Eq. (1.2);
(ii) $u \leqslant g$ in $\mathcal{E} \times (0,T]$, $u \leqslant u_0$ in $(\Omega \cup \mathcal{E}) \times \{0\}$.

*Supersolutions* and *solutions* of (3.2) are defined accordingly.

Our results rely on the following theorems, which are the parabolic counterpart of Theorems 1.1 and 1.2, respectively (see [12] for the proof).

**Theorem 3.3.** *Let assumptions* $(A_1)$, $(A_3)$, $(A_5)$ *and* $(E_1)$ *be satisfied. Suppose* $g \in L^\infty(\mathcal{R} \times (0,T))$, $u_0, c \in L^\infty(\Omega)$. *Let there exist a supersolution* $V$ *of problem* (1.5) *such that* (1.6) *is satisfied. Then there exist infinitely many bounded solutions of problem* (1.4).

**Theorem 3.4.** *Let assumptions* $(A_1)$, $(A_3)$, $(A_5)$ *and* $(E_4)$ *be satisfied. Suppose* $\mathcal{S}_2 \neq \emptyset$, $g \in C([\mathcal{R} \cup \mathcal{S}_1] \times [0, T])$, $u_0 \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$, $g(x, 0) = u_0(x)$ *for any* $x \in \mathcal{R} \cup \mathcal{S}_1$. *Let there exist a subsolution* $Z \leqslant H < 0$ *of problem,*

$$\begin{cases} \mathcal{L}u = \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \mathcal{R}, \end{cases} \tag{3.3}$$

*for some* $\mu \geqslant 0$, *or of problem* (1.5). *Then there exists at most one solution u of problem* (1.8) *such that*

$$\lim_{\operatorname{dist}(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]} |u(x,t)|}{Z(x)} = 0. \tag{3.4}$$

If $\dim \mathcal{S} \leqslant n - 2$, from Theorem 3.4 we obtain the following analogous of Theorem 2.12.

**Theorem 3.5.** *Let assumptions* $(A_1)$, $(A_3)$–$(A_5)$ *and* $(E_4)$ *be satisfied, with* $(A_5)$(ii) *replaced by* $(A_2)$(ii)$'$. *Then*

(i) *there exists at most one solution u of problem* (1.4) *such that*

$$\lim_{d(x,\mathcal{S})\to 0} \frac{\sup_{t\in(0,T]} |u(x,t)|}{\log[d(x,\mathcal{S})]} = 0; \tag{3.5}$$

(ii) *if* $\alpha := \inf_{y\in\mathcal{S}} r(y) - 2 \geqslant 1$, *there exists at most one solution u of problem* (1.4) *such that*

$$\lim_{d(x,\mathcal{S})\to 0} \frac{\sup_{t\in(0,T]} |u(x,t)|}{[d(x,\mathcal{S})]^{-\alpha}} = 0. \tag{3.6}$$

*In particular, problem* (1.4) *has at most one bounded solution.*

If $\dim \mathcal{S} = n - 1$, the following results (to be compared with Theorems 2.16 and 2.20) can be proved.

**Theorem 3.6.** *Let* $\dim \mathcal{S} = n - 1$; *let assumptions* $(A_1)$, $(A_3)$, $(A_5)$ *and* $(E_4)$ *be satisfied. Assume* $\mathcal{S}_2 \neq \emptyset$, $g \in C([\mathcal{R} \cup \mathcal{S}_1] \times [0, T])$, $u_0 \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$, $g(x, 0) = u_0(x)$ *for any* $x \in \mathcal{R} \cup \mathcal{S}_1$. *In addition, let there exist* $\bar{\varepsilon} > 0$ *and a positive, continuous function* $\underline{\rho}$ *satisfying* (2.16)–(2.17). *Then there exists at most one solution u of problem* (1.8) *such that*

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]} |u(x,t)|}{P(d(x,\mathcal{S}_2))} = 0, \tag{3.7}$$

*with P defined in* (2.18).

*In particular, problem* (1.8) *has at most one bounded solution.*

**Theorem 3.7.** *Let* $\dim \mathcal{S} = n - 1$. *Let assumptions* $(A_1)$, $(A_3)$, $(A_5)$ *and* $(E_3)$ *be satisfied. Assume* $\mathcal{S}_2 = \emptyset$, $g \in L^\infty(\mathcal{R} \times (0, T))$, $u_0, c \in L^\infty(\Omega)$. *In addition, let there exist* $\bar{\varepsilon} > 0$ *and a positive continuous function* $\overline{\rho}$ *satisfying* (2.20)–(2.21). *Then infinitely many bounded solutions of problem* (1.4) *exist.*

Let us mention also the parabolic counterpart of Theorems 2.21 and 2.23, respectively.

**Theorem 3.8.** *Let* $\dim \mathcal{S} = n - 1$, $\mathcal{S}_1 \neq \emptyset$ *and* $\mathcal{S}_1 \subseteq \Sigma_2$. *Let assumptions* $(A_1)$, $(A_5)$(iv)–(vi), $(E_1)$, $(F_1)$–$(F_2)$ *be satisfied and* $\sigma_{ij} \in C^1(\overline{\mathcal{S}_1^\varepsilon})$ *for some* $\varepsilon > 0$ $(i, j = 1, \dots, n)$. *In addition, suppose* $c, u_0 \in L^\infty(\Omega)$. *Then infinitely many solutions of problem* (1.4) *exist.*

**Theorem 3.9.** *Let* $\dim \mathcal{S} = n - 1$, $\mathcal{S}_2 \neq \emptyset$ *and* $\mathcal{S}_2 \subseteq \Sigma_1$. *Let assumptions* $(A_1)$, $(A_5)$(iv)–(vi), $(E_4)$, $(F_1)$–$(F_2)$ *be satisfied and* $\sigma_{ij} \in C^1(\overline{\mathcal{S}_2^\varepsilon})$ *for some* $\varepsilon > 0$ $(i, j = 1, \dots, n)$; *moreover, suppose* $g \in C([\mathcal{R} \cup \mathcal{S}_1] \times [0, T])$, $u_0 \in C(\Omega \cup \mathcal{R} \cup \mathcal{S}_1)$, $u_0(x) = g(x, 0)$ *for any* $x \in \mathcal{R} \cup \mathcal{S}_1$. *Then there exists at most one solution of problem* (1.8) *satisfying*

$$\lim_{d(x,\mathcal{S}_2)\to 0} \frac{\sup_{t\in(0,T]} |u(x,t)|}{[d(x,\mathcal{S}_2)]^{-\alpha}} = 0, \tag{3.8}$$

*for some* $\alpha > 0$. *In particular, there exists at most one bounded solution of problem* (1.8).

**Remark 3.10.** In Theorem 2.23 the parameter $\alpha > 0$ must be sufficiently small, while in the last theorem it is arbitrary. This depends on the fact that to prove the former a subsolution of problem (1.9) must be constructed, while for the latter a subsolution of problem (3.3) for some $\mu \geqslant 0$ is needed.

The proofs of Theorems 3.5–3.9 are the same of those given below for the elliptic case, with obvious changes.

## 4. Singular manifolds of low dimension: Proofs

This section is devoted to the proof of Theorem 2.12. As already said, this follows from Theorem 1.2 if we exhibit a suitable subsolution $Z$ of problem (1.9). The construction of $Z$ is rather technical and lengthy; it requires the following steps:

(a) first we construct a local subsolution of the equation $\mathcal{L}u = 0$—namely, for any $\hat{y} \in \mathcal{S}$ we construct a subsolution $z_0$ in $B_{\hat{R}}(\hat{y}) \cap \Omega$, where $B_{\hat{R}}(\hat{y})$ is a ball of radius $\hat{R} > 0$ sufficiently small;
(b) using the compactness of $\mathcal{S}$ and (a) above, we construct a subsolution $z$ of the same equation in a neighborhood $\mathcal{S}^{\varepsilon}$ ($\varepsilon > 0$ sufficiently small);
(c) finally, we extend the subsolution $z$ in $\mathcal{S}^{\varepsilon}$ to a subsolution $Z$ of problem (1.9) with the desired properties.

### 4.1. Technical preliminaries

In view of the compactness and regularity of $\mathcal{S}$ assumed in $(A_1)$(iv), the following holds (see [3] for the proof).

**Lemma 4.1.** *Let assumption $(A_1)$ be satisfied. Then there exists $\sigma > 0$ with the following properties*:

(i) *for any $x \in \mathcal{S}^{\sigma}$ there exists a unique point $x^*(x) \in \mathcal{S}$ such that*

$$d(x, \mathcal{S}) = \left|x - x^*(x)\right|;$$

(ii) $x^*(\cdot) \in C^2(\mathcal{S}^{\sigma}; \mathcal{S}), d(\cdot, \mathcal{S}) \in C^3(\mathcal{S}^{\sigma})$ *and*

$$\nabla\left[d(x, \mathcal{S})\right]^2 = 2\left[x - x^*(x)\right] \quad \left(x \in \mathcal{S}^{\sigma}\right).$$

Let $y^0 \in \mathcal{S}$; let $T_{y^0}\mathcal{S}$ and $\perp_{y^0} \mathcal{S}$ denote the tangent, respectively the orthogonal space to $\mathcal{S}$ at $y^0$.
In view of the compactness of $\mathcal{S}$, we can choose possibly smaller $R_i$ in the representation (2.12), such that

$$\left|\frac{\partial^{|\alpha|}\phi^{(i)}}{\partial y_1^{\alpha_1} \dots \partial y_k^{\alpha_k}}\right| \leqslant C_0 \quad \text{in } \overline{B}_{R_i}\left(y_1^i, \dots, y_k^i\right) \quad \left(|\alpha| \leqslant 3;\ i = 1, \dots, N\right), \tag{4.1}$$

for some $C_0 > 0$ ($\alpha \equiv (\alpha_1, \dots, \alpha_k)$ denoting a multiindex).
It is convenient to point out for further reference a few technical observations; this is the content of the following remark.

**Remark 4.2.** Let $\bar{y} \in \mathcal{S}$. Then there exists $i \in \{1, \dots, N\}$ such that

$$\bar{y} = \left(\bar{y}_1, \dots, \bar{y}_k, \phi^{(i)}(\bar{y}_1, \dots, \bar{y}_k)\right).$$

Take the orthonormal vectors in $(A_4)$(iii) and construct a complete basis of orthonormal vectors $\eta^1(\cdot), \dots, \eta^n(\cdot) \in C^2(B_R(\bar{y}_1, \dots, \bar{y}_k))$, where we may assume that, for $\zeta \in B_R(\bar{y}_1, \dots, \bar{y}_k)$ (for some $R = R(i, \bar{y}) > 0$), $\eta^1(\zeta), \dots, \eta^k(\zeta)$ form a basis in the tangential subspace and $\eta^{k+1}(\zeta), \dots, \eta^n(\zeta)$ form a basis in the orthogonal subspace to $\mathcal{S}$ at $(\zeta, \phi^{(i)}(\zeta))$. In the following we use for simplicity the notation $\eta^l(y)$, $y \in \mathcal{S}$ ($l = 1, \dots, n$) as in $(A_4)$(iii).
The tangent space $T_{\bar{y}}\mathcal{S}$ to $\mathcal{S}$ at $\bar{y}$ can be taken into the linear subspace $\{Y \in \mathbb{R}^n \mid Y_{k+1} = \dots = Y_n = 0\}$ by a transformation of coordinates $Y := M(\bar{y})(y - \bar{y})$, where the rotation matrix is given by $M(\bar{y}) = (\eta^1(\bar{y}) \dots \eta^n(\bar{y}))^T \in \mathcal{M}_n$. This matrix valued function belongs to $C^2$ in some set $B_R(\bar{y}) \cap \mathcal{S}$, i.e., its composition with the local representation

of $\mathcal{S}$ is $C^2(\overline{B}_R(\bar{y}_1,\dots,\bar{y}_k))$. Let the $C^3$ functions $p = p^{(i,\bar{y})} : \overline{U^{(i)}_{\bar{y}}} \to \mathbb{R}^{n-k}$, $p \equiv (p_{k+1},\dots,p_n)(Y_1,\dots,Y_k)$ give a local representation of $\mathcal{S}$ in the closure of a bounded neighborhood of 0, say in $\overline{U^{(i)}_{\bar{y}}} \subset \mathbb{R}^k$. Then,

$$(Y_1,\dots,Y_k,p_{k+1},\dots,p_n) = \overline{M}^{(i)}\big((y_1,\dots,y_k,\phi^i_{k+1},\dots,\phi^i_n) - \bar{y}\big),$$

$p_{k+1},\dots,p_n$ being evaluated at $(Y_1,\dots,Y_k)$ and $(\phi^i_{k+1},\dots,\phi^i_n)$ at $(y_1,\dots,y_k)$. Moreover, since we compose regular functions in compact sets, there exists $C_1 > 0$ such that

$$\left|\frac{\partial^{|\alpha|} p^{(i,\bar{y})}}{\partial Y_1^{\alpha_1}\dots\partial Y_k^{\alpha_k}}\right| \leqslant C_1 \quad \text{in } \overline{U^{(i)}_{\bar{y}}} \quad \big(|\alpha| \leqslant 3;\ i = 1,\dots,N;\ \bar{y} \in \mathcal{S}\big). \tag{4.2}$$

In fact, by a change of $\bar{y}$ the tangent space $T_{\bar{y}}\mathcal{S}$, the $Y$-coordinates and functions $p^{(i,\bar{y})}$ also change; however, inequality (4.2) holds true with a suitable choice of the constant $C_1$ independent from $\bar{y}$, $i$ and $\alpha$.

Let $\sigma > 0$ as in Lemma 4.1 and $x^0 \in \mathcal{S}^\sigma$ be fixed; then the projection $x^*(x^0) \in \mathcal{S}$ is well defined and there exists $i \in \{1,\dots,N\}$ such that

$$x^*\big(x^0\big) = \big(\bar{y}_1,\dots,\bar{y}_k,\phi^{(i)}(\bar{y}_1,\dots,\bar{y}_k)\big).$$

With the notations of the above remark, let $p^{(i,x^*(x^0))}$ be the local representation of $\mathcal{S}$ in the neighborhood $\overline{U}_0 \equiv \overline{U^{(i)}_{x^*(x^0)}}$ of 0 in $\mathbb{R}^k$. As we make below, the new coordinate system $X \equiv (X_1,\dots,X_n)$ can be chosen in $\mathbb{R}^n$ so that, if $p : \overline{U}_0 \to \mathbb{R}^{n-k}$ denotes the local representation of $\mathcal{S}$ with respect to this system, the following holds:

$$(C)\quad \begin{cases} \text{(i)}\ \ X^*(X^0) = 0; \\ \text{(ii)}\ \ \perp_0 \mathcal{S} = \{X \in \mathbb{R}^n \mid X_1 = \dots = X_k = 0\}; \\ \text{(iii)}\ \ X^0 \equiv (0,\dots,0,X^0_n),\ d(X^0,\mathcal{S}) = X^0_n; \\ \text{(iv)}\ \ \frac{\partial^2 p_n}{\partial X_i \partial X_j}(0) = p_n^{ii}\delta_{ij}\ (i,j = 1,\dots,k); \end{cases}$$

here $X^0$, $X^*(X^0)$ denote the new coordinates of the points $x^0$, $x^*(x^0)$. In fact, equalities (i) and (ii) also hold in the $Y$-coordinates, whereas (iii) can be obtained up to a rotation in the orthogonal space $\{Y \in \mathbb{R}^n \mid Y_1 = \dots = Y_k = 0\}$ and (iv) by a rotation in the tangent space $\{Y \in \mathbb{R}^n \mid Y_{k+1} = \dots = Y_n = 0\}$. Hence we have the analogous of inequality (4.2) for some constant $C_2 > 0$ independent of $x^0 \in \mathcal{S}^\sigma$, namely

$$\left|\frac{\partial^{|\alpha|} p_s}{\partial X_1^{\alpha_1}\dots\partial X_k^{\alpha_k}}\right| \leqslant C_2 \quad \text{in } \overline{U}_0 \quad \big(|\alpha| \leqslant 3;\ s = k+1,\dots,n\big). \tag{4.3}$$

In the following we set:

$$p_s^l \equiv \frac{\partial p_s}{\partial X_l}(0), \qquad p_s^{lq} \equiv \frac{\partial^2 p_s}{\partial X_l \partial X_q}(0), \tag{4.4}$$

and so on. Then the choice $(C)$ implies

$$p_s(0) = 0, \qquad p_s^l = 0 \quad (s = k+1,\dots,n;\ l = 1,\dots,k). \tag{4.5}$$

The following lemma deals with derivatives of the projection map $X^*(\cdot)$, respectively of the distance $d(\cdot,\mathcal{S})$. Part of its proof was given in [9]; we reproduce it here for convenience of the reader.

**Lemma 4.3.** *Let assumption $(A_1)$ be satisfied. There exists $\varepsilon_0$ such that, if $\varepsilon \in (0,\varepsilon_0)$, $x^0 \in \mathcal{S}^\varepsilon$ is fixed and the choice $(C)$ is made, the following holds*:

(i) *for any $i = 1,\dots,n$*

$$\frac{\partial X_l^*}{\partial X_i}\big(X^0\big) = \begin{cases} \frac{\delta_{il}}{1 - X_n^0 p_n^{ll}} & \textit{if } l = 1,\dots,k, \\ 0 & \textit{if } l = k+1,\dots,n; \end{cases} \tag{4.6}$$

(ii) *for any* $i, j = 1, \dots, n$,

$$\frac{\partial^2 X_l^*}{\partial X_i \partial X_j}(X^0) = \frac{1}{1 - X_n^0 p_n^{ll}} \left\{ \sum_{s=k+1}^{n} \sum_{q=1}^{k} p_s^{lq} \frac{\delta_{is}\delta_{jq} + \delta_{iq}\delta_{js}}{1 - X_n^0 p_n^{qq}} \right.$$
$$\left. + p_n^{ijl} \frac{X_n^0 \sum_{m,q=1}^{k} \delta_{im}\delta_{jq}}{(1 - X_n^0 p_n^{ii})(1 - X_n^0 p_n^{jj})} \right\} \quad \text{if } l = 1, \dots, k, \tag{4.7}$$

*respectively*

$$\frac{\partial^2 X_l^*}{\partial X_i \partial X_j}(X^0) = \sum_{q,r=1}^{k} p_l^{qr} \frac{\delta_{iq}\delta_{jr}}{(1 - X_n^0 p_n^{ii})(1 - X_n^0 p_n^{jj})}, \quad \text{if } l = k+1, \dots, n; \tag{4.8}$$

(iii) *for any* $i = 1, \dots, n$

$$\left.\frac{\partial d(X, \mathcal{S})}{\partial X_i}\right|_{X = X^0} = \delta_{in}; \tag{4.9}$$

(iv) *there holds*:

$$\left.\frac{\partial^2 d(X, \mathcal{S})}{\partial X_i \partial X_j}\right|_{X = X^0} = \begin{cases} -\frac{p_n^{ii}}{1 - d(X^0, \mathcal{S}) p_n^{ii}} \delta_{ij} & \text{if } i, j = 1, \dots, k, \\ \frac{\delta_{ij} - \delta_{in}\delta_{jn}}{d(X^0, \mathcal{S})} & \text{if } i, j = k+1, \dots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

In the following by $O(|X|), O(|X|^2), \dots$ we denote functions of $X$ such that for some constant $D > 0$,

$$\bigl|O(|X|)\bigr| \leqslant D|X|, \quad O(|X|^2) \leqslant D|X|^2, \quad \dots. \tag{4.11}$$

**Remark 4.4.** Omitting assumptions $(C)$(iii) and $(C)$(iv) in Lemma 4.3 amounts to deal, for any $x^0 \in \mathcal{S}^\varepsilon$, with the $Y$-coordinates mentioned in Remark 4.2 with $\bar{y} = x^*(x^0)$. Hence, if $Y^0, Y, Y^*(Y)$ respectively correspond to $x^0, x, x^*(x)$ in the new coordinates, we have:

$$\begin{gathered} Y^0 \equiv \bigl(0, \dots, 0, Y_{k+1}^0, \dots, Y_n^0\bigr), \qquad Y^*\bigl(Y^0\bigr) = 0, \\ d\bigl(Y^0, \mathcal{S}\bigr) = \bigl|Y^0 - Y^*\bigl(Y^0\bigr)\bigr| = \bigl|Y^0\bigr|. \end{gathered} \tag{4.12}$$

Therefore equalities (4.5) are still valid. From Lemma 4.3, going back from $X$-coordinates to $Y$-coordinates, i.e., performing rotations in the tangential and orthogonal subspaces to $\mathcal{S}$ at $X^0$, gives:

$$\left.\frac{\partial (Y_l - Y_l^*(Y))}{\partial Y_i}\right|_{Y = Y^0} = \delta_{il} - \frac{\partial Y_l^*(Y^0)}{\partial Y_i} = \begin{cases} O(|Y^0|) & \text{if } i, l \leqslant k, \\ \delta_{il} & \text{if } l \geqslant k+1. \end{cases} \tag{4.13}$$

In the above equalities the constant $D$ related to $O(|Y^0|)$ can be chosen independent from $x^0 \in \mathcal{S}^\epsilon$.

Let us introduce for further reference the following notation. For any $i, j = 1, \dots, n$ set $[i, j] := \{i, i+1, \dots, j-1, j\} \subseteq \mathbb{N}$,

$$\chi_{[i,j]}(l) := \begin{cases} 1 & \text{if } i \leqslant l \leqslant j, \\ 0 & \text{otherwise} \end{cases} \qquad (l \in \mathbb{N})$$

(namely, $\chi_{[i,j]}$ is the characteristic function of the string $[i, j]$). Then (4.13) reads:

$$\left.\frac{\partial (Y_l - Y_l^*(Y))}{\partial Y_i}\right|_{Y^0} = \delta_{il} - \frac{\partial Y_l^*}{\partial Y_i}\bigl(Y^0\bigr) = O\bigl(\bigl|Y^0\bigr|\bigr)\chi_{[1,k]}(l)\chi_{[1,k]}(i) + \delta_{il}\chi_{[k+1,n]}(l), \tag{4.14}$$

where we also used (4.12). Likewise,

$$\left.\frac{\partial^2 Y_l^*(Y)}{\partial Y_i \partial Y_j}\right|_{Y^0} = B_{lij}\bigl(Y^0\bigr)\bigl\{\chi_{[1,k]}(l)\bigl(1 - \chi_{[k+1,n]}(i)\chi_{[k+1,n]}(j)\bigr) + \chi_{[k+1,n]}(l)\chi_{[1,k]}(i)\chi_{[1,k]}(j)\bigr\}, \tag{4.15}$$

where the functions $B_{lij}(.)$ are uniformly bounded for $Y^0$ corresponding to any $x^0 \in \mathcal{S}^\varepsilon$.

Now we can prove Lemma 4.3.

**Proof of Lemma 4.3.** Let

$$\varepsilon_0 := \min\left\{\sigma, \frac{1}{2C_2}\right\}, \tag{4.16}$$

where $\sigma > 0$ is given in Lemma 4.1 and $C_2 > 0$ in (4.3). Suppose that $x^0 \in \mathcal{S}^\varepsilon$ ($\varepsilon \in (0, \varepsilon_0)$) is fixed and the choice $(C)$ has been made.

(i) Let $e_i$ be the unit vector of the $i$th coordinate axis ($i = 1, \dots, n$). Since the map $X^*(\cdot)$ is continuous by Lemma 4.1(ii), there exists $\eta > 0$ such that $X^*(X^0 + h_i e_i + h_j e_j) \in \{(X, p(X)) \mid X \in U_0\}$ for any $h_i, h_j \in (-\eta, \eta)$ $(i, j = 1, \dots, n)$. This implies:

$$\left|X^0 + h_i e_i + h_j e_j - X^*\left(X^0 + h_i e_i + h_j e_j\right)\right|^2 = \min_{X \in U_0}\left|X^0 + h_i e_i + h_j e_j - \left(X, p(X)\right)\right|^2,$$

thus

$$\frac{\partial}{\partial X_l}\left|X^0 + h_i e_i + h_j e_j - \left(X, p(X)\right)\right|^2\Big|_{(X, p(X)) = X^*(X^0 + h_i e_i + h_j e_j)} = 0, \tag{4.17}$$

for any $l = 1, \dots, k$ and $i, j, h_i, h_j$ as above.

Let us prove the following:

**Claim.** *For any $i, j = 1, \dots, n$ there exist $\eta' \in (0, \min\{\eta, R_m\})$ and $\varphi \equiv (\varphi_1, \dots, \varphi_k) \in C^2((-\eta', \eta')^2; \mathbb{R}^k)$, $\varphi = \varphi(h_i, h_j)$ such that*

$$X^*\left(X^0 + h_i e_i + h_j e_j\right) = \left(\varphi(h_i, h_j), p\left(\varphi(h_i, h_j)\right)\right), \tag{4.18}$$

*for any $(h_i, h_j) \in (-\eta', \eta')^2$.*

In fact, for any fixed $i, j = 1, \dots, n$ define $F \equiv F^{i,j} : (-\eta, \eta)^2 \times U_0 \to \mathbb{R}^k$, $F \equiv (F_1, \dots, F_k)$ as follows:

$$\begin{aligned} F_l(h_i, h_j, X) &:= \frac{\partial}{\partial X_l}\left|\left(X^0 + h_i e_i + h_j e_j\right) - \left(X, p(X)\right)\right|^2 \\ &= -2(h_i\delta_{il} + h_j\delta_{jl} - X_l) - 2\sum_{s=k+1}^{n}\left(X_s^0 + h_i\delta_{is} + h_j\delta_{js} - p_s(X)\right)\frac{\partial p_s}{\partial X_l}(X). \end{aligned}$$

An elementary calculation gives:

$$\begin{aligned} \frac{\partial F_l}{\partial X_h}(h_i, h_j, X) &= 2\delta_{hl} + 2\sum_{s=k+1}^{n}\left[\frac{\partial p_s}{\partial X_h}(X)\frac{\partial p_s}{\partial X_l}(X) - \left(X_s^0 + h_i\delta_{is} + h_j\delta_{js} - p_s(X)\right)\frac{\partial^2 p_s}{\partial X_h \partial X_l}(X)\right], \\ \frac{\partial F_l}{\partial h_i}(h_i, h_j, X) &= -2\delta_{il} - 2\sum_{s=k+1}^{n}\delta_{is}\frac{\partial p_s}{\partial X_l}(X), \end{aligned}$$

for any $h, l = 1, \dots, k$, $i = 1, \dots, n$. In particular,

$$\frac{\partial F_l}{\partial X_h}(0,0,0) = 2\left(1 - X_n^0 p_n^{hh}\right)\delta_{hl} \quad (h, l = 1, \dots, k), \tag{4.19}$$

$$\frac{\partial F_l}{\partial h_i}(0,0,0) = -2\delta_{il} \quad (l = 1, \dots, k;\ i = 1, \dots, n) \tag{4.20}$$

(here use of $(C)$ and (4.5) has been made).

By (4.16) and (4.3)

$$0 \leqslant X_n^0 \leqslant \varepsilon \leqslant \frac{1}{2C_2} \leqslant \frac{1}{2|p_n^{hh}|},$$

hence from (4.3) and (4.19) we obtain:

$$\left|\frac{\partial(F_1,\dots,F_k)}{\partial(X_1,\dots,X_k)}\right|(0,0,0)\geqslant 1.$$

Then by the Implicit Function Theorem there exists $\eta'\in(0,\eta)$ and uniquely defined functions $\varphi_1,\dots,\varphi_k\in C^2((-\eta',\eta')^2)$ such that $\varphi_l(0,0)=0$ $(l=1,\dots,k)$, and

$$F\big(h_j,h_j,\varphi_1(h_i,h_j),\dots,\varphi_k(h_i,h_j)\big)=0,\tag{4.21}$$

for any $(h_i,h_j)\in(-\eta',\eta')^2$. Thus

$$X_l^*\big(X^0+h_ie_i+h_je_j\big)=\varphi_l(h_i,h_j),$$

for any $s=1,\dots,k$, $(h_i,h_j)\in(-\eta',\eta')^2$. Hence equality (4.18) and the claim follow.

Let $i=1,\dots,n$. From (4.18)–(4.20) we plainly have:

$$\frac{\partial X_l^*}{\partial X_i}\big(X^0\big)=\frac{\partial X_l^*}{\partial h_i}\big(X^0+h_ie_i\big)\bigg|_{h_i=0}=\frac{\partial\varphi_l}{\partial h_i}(0,0)=\frac{\delta_{il}}{1-X_n^0p_n^{ll}},\tag{4.22}$$

if $l=1,\dots,k$, respectively:

$$\frac{\partial X_l^*}{\partial X_i}\big(X^0\big)=\frac{\partial}{\partial h_i}p_l\big(\varphi(h_i,0)\big)\bigg|_{h_i=0}=\sum_{m=1}^k p_l^m\frac{\partial\varphi_m}{\partial h_i}(0,0)=0,\tag{4.23}$$

if $l=k+1,\dots,n$ (here use of (4.5) has been made). Hence equality (4.6) follows.

(ii) In view of (4.18), for any $i,j=1,\dots,n$, $l=1,\dots,k$, we have:

$$\frac{\partial^2}{\partial h_i\partial h_j}F_l\big(h_i,h_j,\varphi(h_i,h_j)\big)\bigg|_{h_i=h_j=0}=0.\tag{4.24}$$

As a lengthy calculation shows, the above equality reads:

$$\begin{aligned}&\frac{\partial^2\varphi_l}{\partial h_i\partial h_j}(0,0)-\sum_{s=k+1}^n\sum_{q=1}^k p_s^{lq}\frac{\delta_{is}\delta_{jq}+\delta_{iq}\delta_{js}}{1-X_n^0p_n^{qq}}-p_n^{ijl}\frac{X_n^0\sum_{m,q=1}^k\delta_{im}\delta_{jq}}{(1-X_n^0p_n^{ii})(1-X_n^0p_n^{jj})}\\&\quad-X_n^0\sum_{q=1}^k p_n^{lq}\frac{\partial^2\varphi_q}{\partial h_i\partial h_j}(0,0)=0;\end{aligned}\tag{4.25}$$

hence by $(C)$(iv) equality (4.7) follows.

In view of (4.18), for any $i,j=1,\dots,n$, $l=k+1,\dots,n$ we have:

$$\frac{\partial^2X_l^*}{\partial X_i\partial X_j}\big(X^0\big)=\frac{\partial^2p_l(\varphi(h_i,h_j))}{\partial h_i\partial h_j}\bigg|_{h_i=h_j=0}=\sum_{q,r=1}^k p_l^{qr}\frac{\delta_{iq}\delta_{jr}}{(1-X_n^0p_n^{qq})(1-X_n^0p_n^{rr})}$$

(here use of (4.5) has been made). Hence (4.8) follows by $(C)$(iv).

(iii) Observe preliminarily that for any $X\in\mathcal{S}^\varepsilon$,

$$d^2(X,\mathcal{S})=\sum_{l=1}^n\big[X_l-X_l^*(X)\big]^2,$$

the projection $X^*(X)\in\mathcal{S}$ being well defined by Lemma 4.1(i).

For any $i=1,\dots,n$,

$$\sum_{l=1}^n\frac{\partial[X_l-X_l^*(X)]^2}{\partial X_i}=2\sum_{l=1}^n\big(X_l-X_l^*(X)\big)\bigg(\delta_{il}-\frac{\partial X_l^*}{\partial X_i}(X)\bigg).\tag{4.26}$$

By $(C)$ there holds $X_l^0 - X_l^*(X^0) = X_n^0\delta_{ln}$ $(l = 1, \dots, n)$, thus we get:

$$\left.\frac{\partial d^2(X,\mathcal{S})}{\partial X_i}\right|_{X=X^0} = 2X_n^0\delta_{in} = 2d\big(X^0,\mathcal{S}\big)\delta_{in}; \tag{4.27}$$

here use of (4.6) has been made. Hence equality (4.9) follows.

(iv) It is easily seen that

$$\frac{\partial^2 d}{\partial X_i \partial X_j} = \frac{1}{2d}\left(\frac{\partial^2 d^2}{\partial X_i \partial X_j} - \frac{1}{2d^2}\frac{\partial d^2}{\partial X_i}\frac{\partial d^2}{\partial X_j}\right), \tag{4.28}$$

for any $X \in \mathcal{S}^\varepsilon$, $i, j = 1, \dots, n$ (here $d \equiv d(X,\mathcal{S})$ for simplicity). From (4.26) we get:

$$\begin{aligned}&\sum_{l=1}^n \frac{\partial^2[X_l - X_l^*(X)]^2}{\partial X_i \partial X_j}\\ &\quad = 2\sum_{l=1}^n\left[\left(\delta_{jl} - \frac{\partial X_l^*}{\partial X_j}(X)\right)\left(\delta_{il} - \frac{\partial X_l^*}{\partial X_i}(X)\right) - \big(X_l - X_l^*(X)\big)\frac{\partial^2 X_l^*}{\partial X_i \partial X_j}(X)\right] =: 2(S_1 - S_2).\end{aligned} \tag{4.29}$$

For $X = X^0$, using the choice $(C)$, (4.6) and (4.8), we obtain easily:

$$S_1 = \begin{cases} \frac{(X_n^0 p_n^{ii})^2}{(1-X_n^0 p_n^{ii})^2}\delta_{ij} & \text{if } i, j = 1, \dots, k,\\ \delta_{ij} & \text{if } i, j = k+1, \dots, n,\\ 0 & \text{otherwise,}\end{cases}$$

$$S_2 = \begin{cases} \frac{X_n^0 p_n^{ii}}{(1-X_n^0 p_n^{ii})^2}\delta_{ij} & \text{if } i, j = 1, \dots, k,\\ 0 & \text{otherwise.}\end{cases}$$

Then from (4.27)–(4.29) equality (4.10) easily follows. This completes the proof. □

### 4.2. *Proof of Theorem 2.12*

*4.2.1.* Let us prove the following proposition, which corresponds to the first step of the construction of the subsolution $Z$.

**Proposition 4.5.** *Let the assumptions of Theorem* 2.12 *be satisfied*; *let $\hat{y} \in \mathcal{S}$ be fixed. Then there exist $\hat{R} > 0$ and $z_0 \in C^2(B_{\hat{R}}(\hat{y}) \cap \Omega)$ such that $z_0$ is a subsolution of the equation*:

$$\mathcal{L}u = 0 \quad \text{in } B_{\hat{R}}(\hat{y}) \cap \Omega. \tag{4.30}$$

*Moreover,*

(i) $\sup_{B_{\hat{R}}(\hat{y})\cap\Omega} z_0 < +\infty$;
(ii) $z_0(x) \sim \log[d(x,\mathcal{S})]$ *as* $d(x,\mathcal{S}) \to 0$ $(x \in B_{\hat{R}}(\hat{y}) \cap \Omega)$.

*If* $\alpha := \inf_{y\in\mathcal{S}} r(y) - 2 \geqslant 1$, *the same conclusion holds with* (ii) *replaced by,*

(iii) $z_0(x) \sim -[d(x,\mathcal{S})]^{-\alpha}$ *as* $d(x,\mathcal{S}) \to 0$ $(x \in B_{\hat{R}}(\hat{y}) \cap \Omega)$.

To prove Proposition 4.5 we have to construct suitable matrix functions. Let $\hat{y} \in \mathcal{S}$ be fixed; take $\hat{R} \leqslant \varepsilon$, with $\varepsilon$ as in Lemma 4.3. Taking possibly smaller $\hat{R}$ we have $\mathcal{B}_{\hat{R}}(\hat{y}) := B_{\hat{R}}(\hat{y}) \cap \mathcal{S} \subseteq \mathcal{U}_i$ $(i \in \{1, \dots, N\}$; see (2.12)). Moreover, by $(A_4)$(ii) there exists $\hat{r} \in \{2, \dots, n-k\}$ such that $\hat{r}$ is a lower bound for the orthogonal rank of $A$ in $\mathcal{S}$.

For any $y \in \mathcal{B}_{\hat{R}}(\hat{y})$ set $A \equiv A(y)$ and, using the notation in Remark 4.2, $M \equiv M(y) \equiv M^{(i)}(y)$; then $M(\cdot) \in C^2(\mathcal{B}_{\hat{R}}(\hat{y}))$. In the new coordinate system $X := M(x - y)$ there holds $X^*(y) = 0$,

$$\perp_0 \mathcal{S} = \big\{X \in \mathbb{R}^n \bigm| X_1 = \dots = X_k = 0\big\}.$$

To obtain a convenient representation of the diffusion matrix $A \equiv (a_{ij})$ in the new system, define:

$$\tilde{A} := MAM^T; \tag{4.31}$$

let $\tilde{\alpha} \in \mathcal{M}_{n-k}$ denote the matrix with entries $(\tilde{A})_{k+i,k+j}$ $(i, j = 1, \dots, n-k)$. It is immediately seen that the rank of the matrix $\tilde{\alpha}$ coincides with the orthogonal rank of the matrix $A$ at $y$. In fact, there holds:

$$(\tilde{A})_{k+i,k+j} = \langle \tilde{A} e_{k+i}, e_{k+j} \rangle = \langle A\eta^{(k+i)}, \eta^{(k+j)} \rangle,$$

here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$, $e_h$ the unit vector of the $h$th coordinate axis and $\eta^{(k+i)} := M^T e_{k+i} \in \perp_y \mathcal{S}$ $(i = 1, \dots, n-k)$ by the choice of $M$. Thus $\tilde{\alpha} = A_\perp$ and the claim clearly follows from Definition 2.11.

Let $\xi^{(k+i)} \in C^2(\mathcal{B}_{\hat{R}}(\hat{y}))$ denote the unit eigenvectors considered in $(A_4)$(iii), $\lambda_{k+i}$ the corresponding eigenvalues of the matrix $\tilde{\alpha}$ $(i = 1, \dots, n-k)$. Recall that $\tilde{\alpha}$ is symmetric and $\tilde{\alpha}(\cdot) \in C^2(\mathcal{B}_{\hat{R}}(\hat{y}); \mathcal{M}_{n-k})$. Then $\lambda_{k+i} = \langle \tilde{\alpha}\xi^{(k+i)}, \xi^{(k+i)} \rangle$ $(i = 1, \dots, n-k)$ are $C^2(\mathcal{B}_{\hat{R}}(\hat{y}))$, too. Set

$$q_1 := \left( \left( \xi_j^{(k+i)} \right) \right)_{i,j=1,\dots,n-k} \in \mathcal{M}_{n-k}$$

(where $\xi^{(k+i)}$ denotes the $i$th row of the matrix $q_1$); clearly, the rank of the matrix $q_1 \tilde{\alpha} q_1^T$ is not less than $\hat{r}$ and there holds:

$$\left( q_1 \tilde{\alpha} q_1^T \right)_{ij} = \lambda_{k+i} \delta_{ij} \quad (i, j = 1, \dots, n-k).$$

Since at least $\hat{r}$ eigenvalues of $\tilde{\alpha}$ are strictly positive and continuous, choosing possibly smaller $\hat{R}$ we may assume that $\lambda_{n-\hat{r}+1}, \dots, \lambda_n$, are strictly positive in $\overline{\mathcal{B}_{\hat{R}}(\hat{y})}$. Let $\beta_1 > 0$ be a lower bound for $\lambda_{n-\hat{r}+1}, \dots, \lambda_n$ and $\beta_2 > 0$ an upper bound for all the eigenvalues of $\tilde{\alpha}$ in $\overline{\mathcal{B}_{\hat{R}}(\hat{y})}$, namely:

$$\beta_1 \leqslant \lambda_{n-\hat{r}+1}, \dots, \lambda_n, \quad \text{and} \quad \lambda_{k+1}, \dots, \lambda_n \leqslant \beta_2, \quad \text{in } \overline{\mathcal{B}_{\hat{R}}(\hat{y})}. \tag{4.32}$$

Define the matrix $q_2 \in \mathcal{M}_{n-k}$ as follows:

$$(q_2)_{ij} = \begin{cases} \frac{1}{\sqrt{\beta_2}} & \text{if } i = j = 1, \dots, n-k-\hat{r}, \\ \frac{1}{\sqrt{\lambda_{k+i}}} & \text{if } i = j = n-k-\hat{r}+1, \dots, n-k, \\ 0 & \text{elsewhere;} \end{cases}$$

set also $q_0 := q_2 q_1$. Then we have:

$$\left( q_0 \tilde{\alpha} q_0^T \right)_{ij} = \begin{cases} \frac{\lambda_{k+i}}{\beta_2} & \text{if } i = j = 1, \dots, n-k-\hat{r}, \\ 1 & \text{if } i = j = n-k-\hat{r}+1, \dots, n-k, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.33}$$

Finally, define the matrix $Q_0 \in \mathcal{M}_n$ as follows:

$$(Q_0)_{ij} := \begin{cases} \delta_{ij} & \text{if } i, j = 1, \dots, k, \\ (q_0)_{i-k,j-k} & \text{if } i, j = k+1, \dots, n, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.34}$$

Then by (4.33) we have:

$$\left( Q_0 \tilde{A} Q_0^T \right)_{ij} = \begin{cases} \frac{\lambda_i}{\beta_2} & \text{if } i = j = k+1, \dots, n-\hat{r}, \\ \delta_{ij} & \text{elsewhere for } i, j = k+1, \dots, n; \end{cases} \tag{4.35}$$

moreover, the matrix function $Q_0$ belongs to $C^2(\mathcal{B}_{\hat{R}}(\hat{y}); \mathcal{M}_n)$.

Now we can prove Proposition 4.5.

**Proof of Proposition 4.5.** (i) Let us first prove the result assuming $\beta = 0$ in $(A_2)$(ii)$'$, namely $b_i \in L^\infty(\Omega)$ $(i = 1, \dots, n)$. Fix $\hat{y} \in \mathcal{S}$; let $\hat{R} \in (0, \varepsilon)$ be as required in the above construction. For any $y \in \mathcal{B}_{\hat{R}}(\hat{y})$ set

$$E(y) := \left[ M(y) \right]^T Q_0(y) M(y),$$

$B_{\hat{R}} \equiv B_{\hat{R}}(\hat{y})$ and, for any $x \in B_{\hat{R}} \cap \Omega$,

$$z_0(x) := \log\bigl|E\bigl(x^*(x)\bigr)\bigl[x - x^*(x)\bigr]\bigr|^2 + K\bigl|E\bigl(x^*(x)\bigr)\bigl[x - x^*(x)\bigr]\bigr|, \tag{4.36}$$

with $K > 0$ to be chosen. Since $E \in C^2(\mathcal{B}_{\hat{R}}(\hat{y}); \mathcal{M}_n)$ by the above remarks and $x^* \in C^2(\mathcal{S}^\sigma; \mathcal{S})$ by Lemma 4.1(ii), we have $z_0 \in C^2(B_{\hat{R}} \cap \Omega)$. Clearly, $z_0$ satisfies (i)–(ii) (observe that $d(x, \mathcal{S}) = |x - x^*(x)|$ and (4.32)–(4.34) hold). The conclusion will follow, if we prove that

$$(\mathcal{M}z_0)\bigl(x^0\bigr) \geqslant 0 \quad \text{for any } x^0 \in B_{\hat{R}} \cap \Omega. \tag{4.37}$$

To this purpose, for any $x^0 \in B_{\hat{R}} \cap \Omega$ a change of variables as in Remark 4.2 is expedient. As above, denote by $X \equiv (X_1, \dots, X_n)$ the new coordinate system, where

$$X := M(\bar{y})(x - \bar{y}), \quad \bar{y} := x^*\bigl(x^0\bigr). \tag{4.38}$$

Set $\overline{M} \equiv M(\bar{y})$; then

$$X^0 := \overline{M}\bigl(x^0 - \bar{y}\bigr) \equiv \bigl(0, \dots, 0, X^0_{k+1}, \dots, X^0_n\bigr), \quad d\bigl(X^0, \mathcal{S}\bigr) = \bigl|X^0\bigr|. \tag{4.39}$$

Define also,

$$Z_0(X) := z_0(x) \quad (x \in B_{\hat{R}} \cap \Omega);$$

it is easily seen that

$$Z_0(X) = \log\bigl|Q\bigl(X^*(X)\bigr)\bigl(X - X^*\bigr)\bigr|^2 + K\bigl|Q\bigl(X^*(X)\bigr)\bigl(X - X^*\bigr)\bigr|, \tag{4.40}$$

for any $x \in B_{\hat{R}} \cap \Omega$, where

$$\begin{gathered} X^*(X) := \overline{M}\bigl(x^*(x) - \bar{y}\bigr), \\ Q\bigl(X^*(X)\bigr) := \overline{M}E\bigl(x^*(x)\bigr)\overline{M}^T = \overline{M}\bigl[M(x^*(x)\bigr]^T Q_0\bigl(x^*(x)\bigr)M\bigl(x^*(x)\bigr)\overline{M}^T. \end{gathered} \tag{4.41}$$

For simplicity we always write $X^* \equiv X^*(X)$, omitting the dependence on $X$; we also set:

$$Q(X^*) \equiv \bigl((q_{ij})\bigr) \quad (x \in B_{\hat{R}} \cap \Omega;\ i, j = 1, \dots, n), \tag{4.42}$$

In particular for $x = x^0$, by (4.41), we get:

$$Q(0) = Q_0(\bar{y}). \tag{4.43}$$

Moreover, we set:

$$W_i = W_i(X) := \sum_{j=1}^n q_{ij}\bigl(X_j - X^*_j\bigr) \quad (i = 1, \dots, n),$$

thus

$$\bigl|Q(X^*)(X - X^*)\bigr| = \left(\sum_{i=1}^n W_i^2\right)^{1/2}. \tag{4.44}$$

It is also easily seen that

$$(\mathcal{M}z_0)\bigl(x^0\bigr) = (\tilde{\mathcal{M}}Z_0)\bigl(X^0\bigr) \quad \bigl(x^0 \in B_{\hat{R}} \cap \Omega\bigr), \tag{4.45}$$

where by $\tilde{\mathcal{M}}$ we denote the formal differential operator:

$$\begin{gathered} \tilde{\mathcal{M}}u \equiv \sum_{i,j=1}^n \tilde{a}_{ij}(X)\frac{\partial^2 u}{\partial X_i \partial X_j} + \sum_{i=1}^n \tilde{b}_i(X)\frac{\partial u}{\partial X_i}, \\ \tilde{a}_{ij}(X) = \tilde{a}_{ji}(X) := \sum_{k.l=1}^n (\overline{M})_{ik} a_{ij}(x)(\overline{M})_{jl} \quad (i, j = 1, \dots, n), \\ \tilde{b}_i(X) := \sum_{j=1}^n (\overline{M})_{ij} b_j(x) \quad (i = 1, \dots, n). \end{gathered} \tag{4.46}$$

Observe that, setting $\tilde{A} \equiv ((\tilde{a}_{ij}))_{i,j=1,\dots,n}$, $\tilde{b} \equiv (\tilde{b}_i)_{i=1,\dots,n}$, equality (4.46) reads:

$$\tilde{A}(X) := \overline{M}A(x)\overline{M}^T, \qquad \tilde{b}(X) := \overline{M}b(x)$$

which coincides with (4.31) if $x = \bar{y}$, $X = 0$. Now we prove that

$$(\tilde{\mathcal{M}}Z_0)\bigl(X^0\bigr) \geqslant 0 \quad \text{for any } x^0 \in B_{\hat{R}} \cap \Omega, \tag{4.47}$$

whence the conclusion will follow.

For $h = 1, \dots, n$, we have:

$$\frac{\partial Z_0}{\partial X_h}(X) = \left\{\frac{2}{|Q(X^*)(X - X^*)|^2} + \frac{K}{|Q(X^*)(X - X^*)|}\right\} I_h^{(1)}, \tag{4.48}$$

where

$$\begin{aligned} I_h^{(1)} &:= \sum_{i=1}^n W_i \frac{\partial W_i}{\partial X_h}, \\ \frac{\partial W_i}{\partial X_h}(X) &= \sum_{j=1}^n \left[\frac{\partial q_{ij}}{\partial X_h}\bigl(X_j - X_j^*\bigr) + q_{ij}\left(\delta_{jh} - \frac{\partial X_j^*}{\partial X_h}\right)\right]. \end{aligned} \tag{4.49}$$

For $h, l = 1, \dots, n$, there holds

$$\begin{aligned} \frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X) = {} & \left\{\frac{2}{|Q(X^*)(X - X^*)|^2} + \frac{K}{|Q(X^*)(X - X^*)|}\right\} I_{hl}^{(2)} \\ & - \left\{\frac{4}{|Q(X^*)(X - X^*)|^4} + \frac{K}{|Q(X^*)(X - X^*)|^3}\right\} I_h^{(1)} I_l^{(1)}, \end{aligned} \tag{4.50}$$

where

$$\begin{aligned} I_{hl}^{(2)} &:= \frac{\partial I_h^{(1)}}{\partial X_l} = \sum_{i=1}^n \left[\frac{\partial W_i}{\partial X_h}\frac{\partial W_i}{\partial X_l} + W_i \frac{\partial^2 W_i}{\partial X_h \partial X_l}\right], \\ \frac{\partial^2 W_i}{\partial X_h \partial X_l}(X) &= \sum_{j=1}^n \left[\frac{\partial^2 q_{ij}}{\partial X_h \partial X_l}\bigl(X_j - X_j^*\bigr) + \frac{\partial q_{ij}}{\partial X_h}\left(\delta_{jl} - \frac{\partial X_j^*}{\partial X_l}\right) + \frac{\partial q_{ij}}{\partial X_l}\left(\delta_{jh} - \frac{\partial X_j^*}{\partial X_h}\right) - q_{ij}\frac{\partial^2 X_j^*}{\partial X_h \partial X_l}\right]. \end{aligned} \tag{4.51}$$

To prove (4.47) the above quantities must be calculated at $X = X^0$, hence (4.43) and (4.42) must be used.

In the following we use the notation $O(|X^0|)$, $O(|X^0|^2), \dots$ introduced above; the same symbol $O(\cdot)$ will denote any function satisfying (4.11). The constant $D$, although not explicitly given, will not depend on the specific choice of $x^0 \in B_{\hat{R}} \cap \Omega$. In fact, it will only depend on the following quantities:

($\alpha$) the functions $q_{ij}(\cdot)$ (hence on the eigenvalues and eigenvectors of the matrix $\tilde{a}(\cdot)$) and their first and second derivatives in $\overline{B_{\hat{R}} \cap \Omega}$;

($\beta$) the function $p(\cdot)$ which gives a local representation of $\mathcal{S}$ in $\overline{B_{\hat{R}}(\hat{y})}$, and its first and second derivatives;

($\gamma$) the lower and upper bounds $\beta_1, \beta_2$ of the eigenvalues of the orthogonal matrix.

Similar remarks hold for the positive constant $H$ encountered below (see (4.60) and the following formulas).

(a) In view of (4.41), (4.43) and (4.34), we have:

$$W_i\bigl(X^0\bigr) = \sum_{j=k+1}^n q_{ij}(0) X_j^0 \quad (i = 1, \dots, n), \tag{4.52}$$

and

$$\bigl|Q(X^*)(X - X^*)\bigr|\big|_{X = X^0} = \bigl|Q(0)X^0\bigr|. \tag{4.53}$$

(b) From (4.49) and (4.13) we obtain:

$$\frac{\partial W_i}{\partial X_h}(X^0) = \sum_{j=k+1}^{n} \frac{\partial q_{in}}{\partial X_h}(0) X_j^0 + \sum_{j=1}^{k} q_{ij}(0) O(|X^0|) + \sum_{j=k+1}^{n} q_{ij}(0)\delta_{jh} = q_{ih}(0)\chi_{[k+1,n]}(h) + O(|X^0|). \tag{4.54}$$

Equalities (4.52) and (4.54), together with (4.42), (4.34) give:

$$I_h^{(1)}(X^0) = \sum_{i=1}^{n}\left(\sum_{j=k+1}^{n} q_{ij}(0)X_j^0\right) q_{ih}(0)\chi_{[k+1,n]}(h) + O(|X^0|^2) = \sum_{i,j=k+1}^{n} q_{ij}(0)X_j^0 q_{ih}(0) + O(|X^0|^2). \tag{4.55}$$

Concerning the above equality, observe that $q_{ih}(0) = 0$ for $i = 1, \ldots, k$ and $h \geqslant k+1$.

(c) The formulas in Remark 4.4 and equality (4.51) imply that $\frac{\partial^2 W_i}{\partial X_h \partial X_l}(X)$ is bounded independently from $i, h, l$ and the choice of $x^0 \in B_{\hat{R}} \cap \Omega$. Then (4.52) and (4.54) entail:

$$I_{hl}^{(2)}(X^0) = \sum_{i=k+1}^{n} q_{ih}(0)q_{il}(0) + O(|X^0|). \tag{4.56}$$

From (4.55) we also obtain:

$$\left(I_h^{(1)} I_l^{(1)}\right)(X^0) = \left(\sum_{i,j,r,s=k+1}^{n} q_{ij}(0)X_j^0 q_{ih}(0) q_{rs}(0) X_s^0 q_{rl}(0)\right) + O(|X^0|^3). \tag{4.57}$$

To prove (4.37) we need an estimate from below of the quantity,

$$\sum_{h,l=1}^{n} \tilde{a}_{hl}(0) \frac{\partial^2 Z_0}{\partial X_h \partial X_l}(X^0) \tag{4.58}$$

(see (4.60)). To this aim, the following computations will be needed, where (4.33), (4.34) and (4.42) are used:

$$\sum_{h,l=1}^{n} \sum_{i=k+1}^{n} \tilde{a}_{hl}(0) q_{ih}(0) q_{il}(0) = \sum_{i=k+1}^{n} \sum_{h,l=k+1}^{n} \tilde{a}_{hl}(0) q_{ih}(0) q_{il}(0) = \sum_{j=1}^{n-k} \sum_{s,t=1}^{n-k} (q_0)_{js} \tilde{a}_{s+k,t+k}(0)(q_0)_{jt} = \sum_{j=1}^{n-k} \left(q_0 \tilde{\alpha} q_0^T\right)_{jj} \geqslant \hat{r};$$

here $\hat{r}$ is the lower bound for the orthogonal rank of $A$ in $\mathcal{S}$ used in the construction of the matrix $Q_0$. It is similarly seen that (see (4.32)):

$$\begin{aligned}
&\sum_{h,l=1}^{n} \tilde{a}_{hl}(0) \sum_{i,j,s,t=k+1}^{n} q_{ij}(0) X_j^0 q_{ih}(0) q_{st}(0) X_t^0 q_{sl}(0) \\
&= \sum_{i,j,s,t=k+1}^{n} q_{ij}(0) X_j^0 q_{st}(0) X_t^0 \left[\sum_{h,l=k+1}^{n} q_{ih}(0) \tilde{a}_{hl}(0) q_{sl}(0)\right] \\
&= \sum_{i,s=k+1}^{n} \left(Q(0)X^0\right)_i \left(Q(0)X^0\right)_s \left(q_0 \tilde{\alpha} q_0^T\right)_{i-ks-k} \\
&= \sum_{i=k+1}^{n-\hat{r}} \frac{\lambda_i}{\beta_2} \left(Q(0)X^0\right)_i^2 + \sum_{i=n-\hat{r}+1}^{n} \left(Q(0)X^0\right)_i^2 \leqslant \left|Q(0)X^0\right|^2.
\end{aligned}$$

Further observe that

$$\frac{|X^0|}{\sqrt{\beta_2}} \leqslant \left|Q(0)X^0\right| \leqslant \frac{|X^0|}{\sqrt{\beta_1}}. \tag{4.59}$$

In view of equalities (4.50), (4.53), (4.56), (4.57) and the above computations, since $\hat{r} \geqslant 2$ by assumption $(A_4)$(ii), we obtain:

$$\begin{aligned}\sum_{h,l=1}^{n} \tilde{a}_{hl}(0)\frac{\partial^2 Z_0}{\partial X_h \partial X_l}\left(X^0\right) &\geqslant \frac{1}{|Q(0)X^0|^2}\left\{2\hat{r} + O\left(\left|X^0\right|\right) + K\left|Q(0)X^0\right|(\hat{r}-1) - 4 + K\,O\left(\left|X^0\right|^2\right)\right\} \\ &\geqslant \frac{1}{|Q(0)X^0|}\left\{-H + K + K\,O\left(\left|X^0\right|\right)\right\}.\end{aligned} \tag{4.60}$$

Here $H$ is a positive constant, independent on $x^0 \in B_{\hat{R}} \cap \Omega$, such that

$$\left|O\left(\left|X^0\right|\right)\right| \leqslant H\left|Q(0)X^0\right|,$$

and $K > 0$ is the constant in the definition of $z_0$ to be chosen (see (4.36)). We also have:

$$\left|\frac{\partial^2 Z_0}{\partial X_h \partial X_l}\left(X^0\right)\right| \leqslant \frac{1}{|Q(0)X^0|^2}\left\{H + K\,O\left(\left|X^0\right|\right)\right\}. \tag{4.61}$$

Since $\tilde{b}_i$ and $q_{ij}$ are uniformly bounded for $x^0 \in B_{\hat{R}} \cap \Omega$, by (4.48), (4.55) we get:

$$\left|\sum_{i=1}^{n} \tilde{b}_i\left(X^0\right)\frac{\partial Z_0}{\partial X_i}\left(X^0\right)\right| \leqslant \frac{1}{|Q(0)X^0|}\left\{H + K\left|O\left(\left|X^0\right|\right)\right|\right\}. \tag{4.62}$$

Inequalities (4.60)–(4.62) imply:

$$\begin{aligned}(\tilde{\mathcal{M}}Z_0)\left(X^0\right) &= \sum_{h,l=1}^{n}\left[\tilde{a}_{hl}\left(X^0\right) - \tilde{a}_{hl}(0)\right]\frac{\partial^2 Z_0}{\partial X_h \partial X_l}\left(X^0\right) \\ &\quad + \sum_{h,l=1}^{n} \tilde{a}_{hl}(0)\frac{\partial^2 Z_0}{\partial X_h \partial X_l}\left(X^0\right) + \sum_{i=1}^{n}\tilde{b}_i\left(X^0\right)\frac{\partial Z_0}{\partial X_i}\left(X^0\right) \\ &\geqslant \frac{1}{|Q(0)X^0|}\left\{-H + K - K\left|O\left(\left|X^0\right|\right)\right|\right\},\end{aligned} \tag{4.63}$$

for the coefficients $\tilde{a}_{hl}$ are of class $C^1$, thus locally Lipschitz continuous in $B_{\hat{R}} \cap \Omega$. For a sufficiently small $\hat{R}$ we have $|O(|X^0|)| < \frac{1}{2}$, then (see (4.59)),

$$(\tilde{\mathcal{M}}Z_0)\left(X^0\right) \geqslant \frac{\sqrt{\beta_1}}{2|X^0|}(K - 2H) > 0, \tag{4.64}$$

choosing $K > 2H$.

Finally, instead of definition (4.36) set:

$$z_0(x) := -\frac{1}{|E(x^*(x))(x - x^*(x))|} + K\log\left|E\left(x^*(x)\right)\left(x - x^*(x)\right)\right|^2, \tag{4.65}$$

if $\inf_{y\in\mathcal{S}} r(y) \geqslant 3$, respectively

$$z_0(x) := -\frac{1}{|E(x^*(x))(x - x^*(x))|^{\alpha}} - \frac{K}{|E(x^*(x))(x - x^*(x))|^{\alpha-1}}, \tag{4.66}$$

if $\inf_{y\in\mathcal{S}} r(y) \geqslant 4$. Arguing as in the case $\inf_{y\in\mathcal{S}} r(y) \geqslant 2$, inequality (4.47) is seen to hold in this cases, too; we omit the details. The proof for the case $\beta = 0$ is complete.

(ii) Now we assume $\beta \in (0,1)$ in $(A_2)$(ii)$'$. In this case the definition (4.36) of $z_0$ is replaced by:

$$z_0(x) := \log\left|E\left(x^*(x)\right)\left(x - x^*(x)\right)\right|^2 + K\left|E\left(x^*(x)\right)\left(x - x^*(x)\right)\right|^{1-\beta}. \tag{4.67}$$

Then inequalities (4.60)–(4.63) are replaced by:

$$\sum_{h,l=1}^{n} \tilde{a}_{hl}(0)\frac{\partial^2 Z_0}{\partial X_h \partial X_l}\big(X^0\big) \geqslant \frac{1}{|Q(0)X^0|}\left\{-H + \frac{K(1-\beta)(\hat{r}-1-\beta)}{|Q(0)X^0|^{\beta}} + KO\big(\big|X^0\big|^{1-\beta}\big)\right\}, \tag{4.68}$$

$$\left|\frac{\partial^2 Z_0}{\partial X_h \partial X_l}\big(X^0\big)\right| \leqslant \frac{1}{|Q(0)X^0|^2}\big\{H + KO\big(\big|X^0\big|^{1-\beta}\big)\big\}. \tag{4.69}$$

Similarly, recalling also $(A_2)(\mathrm{ii})'$ instead of (4.62) we get:

$$\left|\sum_{i=1}^{n} \tilde{b}_i\big(X^0\big)\frac{\partial Z_0}{\partial X_i}\big(X^0\big)\right| \leqslant \frac{1}{|Q(0)X^0|^{1+\beta}}\big\{H + K\big|O\big(\big|X^0\big|^{1-\beta}\big)\big|\big\}. \tag{4.70}$$

Therefore we have in the present case,

$$\begin{aligned}(\tilde{\mathcal{M}} Z_0)\big(X^0\big) &= \sum_{h,l=1}^{n}\big[\tilde{a}_{hl}\big(X^0\big) - \tilde{a}_{hl}(0)\big]\frac{\partial^2 Z_0}{\partial X_h \partial X_l}\big(X^0\big) \\ &\quad + \sum_{h,l=1}^{n} \tilde{a}_{hl}(0)\frac{\partial^2 Z_0}{\partial X_h \partial X_l}\big(X^0\big) + \sum_{i=1}^{n} \tilde{b}_i\big(X^0\big)\frac{\partial Z_0}{\partial X_i}\big(X^0\big) \\ &\geqslant \frac{1}{|Q(0)X^0|^{1+\beta}}\big\{-H + K - K\big|O\big(\big|X^0\big|^{1-\beta}\big)\big|\big\},\end{aligned} \tag{4.71}$$

which corresponds to (4.63). Hence the conclusion follows as before, choosing $\hat{R}$ possibly smaller and $K$ sufficiently large.

Finally, if $\inf_{y\in\mathcal{S}} r(y) \geqslant 3$, instead of definition (4.36) set:

$$z_0(x) := -\frac{1}{|E(x^*(x))(x - x^*(x))|} - \frac{K}{|E(x^*(x))(x - x^*(x))|^{\beta}}, \tag{4.72}$$

or respectively, if $\inf_{y\in\mathcal{S}} r(y) \geqslant 4$,

$$z_0(x) := -\frac{1}{|E(x^*(x))(x - x^*(x))|^{\alpha}} - \frac{K}{|E(x^*(x))(x - x^*(x))|^{\alpha-1+\beta}}. \tag{4.73}$$

Arguing as in the previous cases inequality (4.47) is seen to hold in this cases, too; we omit the details. The proof is complete. □

*4.2.2.* The following proposition corresponds to the second step in the construction of the subsolution $Z$.

**Proposition 4.6.** *Let the assumptions of Theorem* 2.12 *be satisfied. Then there exist* $\varepsilon > 0$ *and* $z \in C^2(\mathcal{S}^{\varepsilon})$ *such that* $z$ *is a subsolution of the equation*:

$$\mathcal{L}u = 0 \quad \text{in } \mathcal{S}^{\varepsilon}. \tag{4.74}$$

*Moreover,*

(i) $\sup_{\mathcal{S}^{\varepsilon}} z < \infty$;
(ii) $z(x) \sim \log[d(x,\mathcal{S})]$ *as* $d(x,\mathcal{S}) \to 0$ $(x \in \mathcal{S}^{\varepsilon})$.

*If* $\alpha := \inf_{y\in\mathcal{S}} r(y) - 2 \geqslant 1$, *the same conclusion holds with* (ii) *replaced by*

(iii) $z(x) \sim -[d(x,\mathcal{S})]^{-\alpha}$ *as* $d(x,\mathcal{S}) \to 0$ $(x \in \mathcal{S}^{\varepsilon})$.

**Proof.** (i) Assume first $\beta = 0$ in assumption $(A_2)(\mathrm{ii})'$. In view of the compactness of $\mathcal{S}$ and of Proposition 4.5, there exist $\hat{y}^i \in \mathcal{S}$, $H_i, h_i, R_i > 0$ and $z_i \in C^2(B_{R_i}(\hat{y}^i) \cap \Omega)$ $(i = 1,\dots,\hat{N}$, for some $\hat{N} \in \mathbb{N})$ of the form:

$$z_i(x) := \log\big|E_i\big(x^*(x)\big)\big[x - x^*(x)\big]\big|^2 + K\big|E_i\big(x^*(x)\big)\big[x - x^*(x)\big]\big|,$$

with the following properties:

(a) $\mathcal{S} \subseteq \bigcup_{i=1}^{\hat{N}} B_{R_i}$;
(b) $z_i$ $(i = 1, \dots, \hat{N})$ is a subsolution of the equation

$$\mathcal{L}u = 0 \quad \text{in } B_{R_i} \cap \Omega,$$

if $K > H_i$ (see (4.64); we set $B_{R_i} \equiv B_{R_i}(\hat{y}^i)$ for simplicity). In fact, there holds:

$$\mathcal{M}z_i \geqslant \frac{h_i}{d(x,\mathcal{S})}(K - H_i) \quad \text{in } B_{R_i} \cap \Omega. \tag{4.75}$$

Observe that the matrix-valued functions $E_i, E_j$ with $i \neq j$ may be different at the same point, since they depend on the local representation of $\mathcal{S}$ and on the choice and ordering of the nonzero eigenvalues.

Choose $\varepsilon \in (0, \min\{R_1, \dots, R_{\hat{N}}\})$, so that $\overline{\mathcal{S}^\varepsilon} \subseteq \bigcup_{i=1}^{\hat{N}} B_{R_i}$. Let $\{\psi_i\}_{i=1}^N$ be a partition of unity subordinate to $\{B_{R_i}\}_{i=1}^N$, namely,

$$\psi_i \in C^\infty(\overline{\Omega}), \quad \operatorname{supp}\psi_i \subseteq B_{R_i}, \qquad 0 \leqslant \psi_i \leqslant 1, \quad \sum_{i=1}^{\hat{N}} \psi_i = 1 \quad \text{in } \overline{\mathcal{S}^\varepsilon}. \tag{4.76}$$

Define:

$$z(x) := \sum_{i=1}^{\hat{N}} \psi_i(x) z_i(x) \quad \big(x \in \mathcal{S}^\varepsilon\big),$$

where we define $z_i := 0$ outside its domain $B_{R_i} \cap \Omega$ $(i = 1, \dots, \hat{N})$. There holds:

$$\mathcal{M}z = \sum_{i=1}^{\hat{N}} (\mathcal{M}\psi_i) z_i + \sum_{i=1}^{\hat{N}} \sum_{h,l=1}^{n} a_{hl} \left( \frac{\partial \psi_i}{\partial x_h} \frac{\partial z_i}{\partial x_l} + \frac{\partial \psi_i}{\partial x_l} \frac{\partial z_i}{\partial x_h} \right) + \sum_{i=1}^{\hat{N}} \psi_i (\mathcal{M}z_i). \tag{4.77}$$

The proof of Proposition 4.5, together with the boundedness in $\mathcal{S}^\varepsilon$ of the functions $p_i, q_{ij}, a_{ij}, b_i$ and their first and second derivatives, easily give the following

**Claim.** *There exist positive constants $C_1, C_2$ such that, taking $\varepsilon > 0$ possibly smaller and $C_3 := \min\{h_1, \dots, h_{\hat{N}}\}$, there holds*:

$$\sum_{i=1}^{\hat{N}} (\mathcal{M}\psi_i) z_i \geqslant C_1 \log\big[d(\cdot,\mathcal{S})\big] \quad \textit{in } \mathcal{S}^\varepsilon; \tag{4.78}$$

$$\sum_{i=1}^{\hat{N}} \sum_{h,l=1}^{n} a_{hl} \left( \frac{\partial \psi_i}{\partial x_h} \frac{\partial z_i}{\partial x_l} + \frac{\partial \psi_i}{\partial x_l} \frac{\partial z_i}{\partial x_h} \right) \geqslant -\frac{C_2}{d(\cdot,\mathcal{S})} \quad \textit{in } \mathcal{S}^\varepsilon; \tag{4.79}$$

$$\sum_{i=1}^{\hat{N}} \psi_i (\mathcal{M}z_i) \geqslant \frac{C_3}{d(\cdot,\mathcal{S})}(K - K_0) \quad \textit{in } \mathcal{S}^\varepsilon, \tag{4.80}$$

*for any $K \geqslant K_0 := \max\{H_1, \dots, H_{\hat{N}}\}$.*

Inequality (4.78) follows from the very definition of $z$, while (4.79), (4.80) are a consequence of (4.48) and (4.55), respectively of (4.75) and (4.76).

From (4.77)–(4.80) we obtain $(x \in \mathcal{S}^\varepsilon)$:

$$(\mathcal{M}z)(x) \geqslant \frac{1}{d(x,\mathcal{S})} \big( C_3 K - C_3 K_0 - C_2 - C_1 d(x,\mathcal{S}) \big|\log\big[d(x,\mathcal{S})\big]\big| \big) \geqslant 0,$$

for any $K$ sufficiently large, thus the result follows in this case. The same argument applies when $\inf_{y\in\mathcal{S}} r(y) \geqslant 3$, whence the conclusion.

(ii) Now suppose $\beta \in (0,1)$ in assumption $(A_2)(\mathrm{ii})'$. Following the proof of Proposition 4.5, we have $z_i$ of the form (4.67); hence inequalities (4.78) and (4.80) become respectively

$$\sum_{i=1}^{\hat N} (\mathcal{M}\psi_i) z_i \geqslant C_1 \frac{\log[d(\cdot,\mathcal{S})]}{d(\cdot,\mathcal{S})^{\beta}} \quad \text{in } \mathcal{S}^{\varepsilon}, \tag{4.81}$$

$$\sum_{i=1}^{\hat N} \psi_i (\mathcal{M} z_i) \geqslant \frac{C_3}{d(\cdot,\mathcal{S})^{1+\beta}} (K - K_0) \quad \text{in } \mathcal{S}^{\varepsilon}. \tag{4.82}$$

Then the conclusion follows as in the previous case for sufficiently large $K$. The case $\inf_{y\in\mathcal{S}} r(y) \geqslant 3$ can be dealt with similarly. This completes the proof. □

*4.2.3.* The third step in the outlined construction of $Z$ allows us to complete the proof of Theorem 2.12.

**Proof of Theorem 2.12.** (i) Let us first address the case when $(E_1)$ holds. Let $z$ be the subsolution of Eq. (4.74) exhibited in Proposition 4.6. Consider the problem:

$$\begin{cases} \mathcal{L}w = 0 & \text{in } \mathcal{S}^{\varepsilon/2}, \\ w = -z & \text{in } \partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}; \end{cases} \tag{4.83}$$

a solution $w \in C^2(\mathcal{S}^{\varepsilon/2}) \cap C^1(\overline{\mathcal{S}^{\varepsilon/2}} \setminus \mathcal{S}) \cap L^{\infty}(\mathcal{S}^{\varepsilon/2})$ is easily constructed by standard compactness arguments. Indeed, observe that any constant $C \geqslant |z|_{L^{\infty}(\partial\mathcal{S}^{\varepsilon/2}\setminus\mathcal{S})}$ is a supersolution of (4.83), whereas $-C$ is a subsolution. Then $\tilde Z := z + w$ is a subsolution of the equation:

$$\mathcal{L}u = 0 \quad \text{in } \mathcal{S}^{\varepsilon/2},$$

such that $\tilde Z = 0$ in $\partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}$ and

$$\tilde Z(x) \sim \log d(x,\mathcal{S}) \quad \text{as } d(x,\mathcal{S}) \to 0.$$

Then $\tilde Z(x) \leqslant 0$ by the maximum principle (applied in sets of the form $\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}^{\delta}$, $\delta \in (0,\varepsilon/2)$ and such that $\tilde Z(x) < 0$ on $\partial\mathcal{S}^{\delta} \cap \mathcal{S}^{\varepsilon/2}$).

Let $W \in C^2(\Omega \setminus \overline{\mathcal{S}^{\varepsilon/2}}) \cap C^1(\Omega \setminus \mathcal{S}^{\varepsilon/2})$ be the solution of the problem:

$$\begin{cases} \mathcal{L}W = 0 & \text{in } \Omega \setminus \overline{\mathcal{S}^{\varepsilon/2}}, \\ W = 1 & \text{on } \mathcal{R}, \\ W = 0 & \text{on } \partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}, \end{cases}$$

hence $0 \leqslant W \leqslant 1$ in $\Omega \setminus \mathcal{S}^{\varepsilon/2}$. Set:

$$Z := \begin{cases} H\tilde Z - 2 & \text{in } \mathcal{S}^{\varepsilon/2}, \\ W - 2 & \text{in } (\Omega \cup \mathcal{R}) \setminus \mathcal{S}^{\varepsilon/2} \end{cases}$$

with $H > 0$ to be chosen. Then:

(a) $Z \in C(\Omega \cup \mathcal{R})$, $Z \leqslant -1$ in $\Omega \cup \mathcal{R}$;
(b) it is easily seen that

$$\int_{\Omega} Z \mathcal{M}^* \psi \, dx \geqslant \int_{\Gamma} \psi \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} (W - H\tilde Z) \nu_j(x) \, dx,$$

for any $\psi \in C_0^{\infty}(\Omega)$, $\psi \geqslant 0$; here $\Gamma := \partial(\mathcal{S}^{\varepsilon/2} \cap \operatorname{supp}\psi)$ and $\nu(x) \equiv (\nu_1(x), \ldots, \nu_n(x))$ denotes the outer normal to $\mathcal{S}^{\varepsilon/2}$ at $x \in \partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}$.

In view of the strong maximum principle, since $\mathcal{L}$ is uniformly elliptic in $\Omega \setminus \overline{\mathcal{S}^{\varepsilon/2}}$ and $\partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S}$ is compact, there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial W}{\partial x_i} \nu_j \geqslant \alpha \quad \text{in } \partial\mathcal{S}^{\varepsilon} \setminus \mathcal{S}.$$

Then choosing $H := \alpha / \{(\sum_{i,j=1}^{n} |a_{ij}|_{L^\infty(\Omega)}) |\nabla \tilde{Z}|_{L^\infty(\partial\mathcal{S}^{\varepsilon/2} \setminus \mathcal{S})}\}$ we obtain:

$$\int_{\Omega} Z \mathcal{M}^* \psi \, dx \geqslant 0.$$

Moreover, by $Z \leqslant -1$, $\rho, \psi, c \geqslant 0$ there holds:

$$\int_{\Omega} Z \mathcal{M}^* \psi \, dx \geqslant 0 \geqslant \int_{\Omega} \rho c Z \psi \, dx.$$

Then $Z$ is a subsolution of (1.9), hence the conclusion follows by Theorem 1.2.

(ii) Now suppose that $(E_2)$ is satisfied and $c(x) > 0$ for any $x \in \Omega$. Let $z$ be the subsolution of Eq. (4.74) exhibited in Proposition 4.6. Since $z$ is defined in $\mathcal{S}^{\varepsilon}$, set $Z := 0$ in $\Omega \setminus \mathcal{S}^{\varepsilon}$. Consider as in (2.9) a function $\chi \in C^2(\overline{\Omega})$, $0 \leqslant \chi \leqslant 1$ such that

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \mathcal{S}^{\varepsilon/2}, \\ 0 & \text{if } x \in \Omega \setminus \mathcal{S}^{\varepsilon}; \end{cases} \tag{4.84}$$

then define $Z(x) := \chi(x) z(x)$, $x \in \Omega$. Clearly, there holds:

$$\mathcal{L}Z \geqslant 0 \quad \text{in } \mathcal{S}^{\varepsilon/2}, \tag{4.85}$$

$$\mathcal{L}Z = 0 \quad \text{in } \Omega \setminus \mathcal{S}^{\varepsilon}, \tag{4.86}$$

$$\mathcal{L}Z(x) \geqslant -K \quad \text{in } \mathcal{S}^{\varepsilon} \setminus \mathcal{S}^{\varepsilon/2}, \tag{4.87}$$

where $K := \max_{\overline{\mathcal{S}^{\varepsilon} \setminus \mathcal{S}^{\varepsilon/2}}} |\mathcal{L}Z|$. Since $Z$ is bounded from above, it is not restrictive to assume,

$$Z \leqslant -\frac{\max\{1, K\}}{c_0} \quad \text{in } \Omega,$$

where $c_0 := \min_{\overline{\mathcal{S}^{\varepsilon} \setminus \mathcal{S}^{\varepsilon/2}}} c > 0$. Hence by inequalities (4.85)–(4.87) we conclude that $Z$ is a subsolution of problem (1.9). Then by Theorem 1.2 the conclusion follows. □

## 5. Singular manifolds of high dimension: Proofs

The present section is devoted to prove the results stated in Sections 2.1 (paragraph 2.1.3) and 2.3.

**Proof of Theorem 2.16.** To prove claim (i) observe that $P(\zeta)$ is a decreasing positive function of $\zeta \in (0, \bar{\varepsilon})$, hence it has a limit as $\zeta \to 0^+$. By contradiction, let this limit be finite. Then for any $\sigma > 0$ there exists $\delta = \delta(\sigma)$ such that for any $\zeta_0, \zeta$ with $0 < \zeta_0 \leqslant \zeta \leqslant \delta$:

$$\begin{aligned} \sigma &\geqslant \int_{\zeta_0}^{\bar{\varepsilon}} (\eta - \zeta_0) \underline{\rho}(\eta) \, d\eta - \int_{\zeta}^{\bar{\varepsilon}} (\eta - \zeta) \underline{\rho}(\eta) \, d\eta \\ &= \int_{\zeta_0}^{\zeta} (\eta - \zeta_0) \underline{\rho}(\eta) \, d\eta + \int_{\zeta}^{\bar{\varepsilon}} (\zeta - \zeta_0) \underline{\rho}(\eta) \, d\eta \geqslant 0. \end{aligned} \tag{5.1}$$

In particular, for any $\zeta_0, \zeta$ as above there holds:

$$0 \leqslant \int_{\zeta}^{\bar{\varepsilon}} (\zeta - \zeta_0) \underline{\rho}(\eta) \, d\eta \leqslant \sigma \quad \big(\zeta \in (0, \delta)\big).$$

As $\zeta_0 \to 0^+$, the above inequality gives:

$$0 \leqslant \zeta \int_{\zeta}^{\bar{\varepsilon}} \underline{\rho}(\eta)\,d\eta \leqslant \sigma \quad \text{for any } \zeta \in (0,\delta). \tag{5.2}$$

Since $\sigma$ is arbitrary and $\delta = \delta(\sigma)$, we get:

$$\lim_{\zeta \to 0^+} \zeta \int_{\zeta}^{\bar{\varepsilon}} \underline{\rho}(\eta)\,d\eta = 0, \tag{5.3}$$

whence by (2.16)

$$\lim_{\zeta \to 0^+} P(\zeta) = \lim_{\zeta \to 0^+} \left[ \int_{\zeta}^{\bar{\varepsilon}} \eta \underline{\rho}(\eta)\,d\eta - \zeta \int_{\zeta}^{\bar{\varepsilon}} \underline{\rho}(\eta)\,d\eta \right] = +\infty, \tag{5.4}$$

a contradiction. This proves the claim.

To prove (ii), set

$$\mathcal{Z}(\zeta) := e^{\frac{C}{\beta}\zeta} - \frac{1}{C} \int_{\zeta}^{\bar{\varepsilon}} \left[1 - e^{\frac{C}{\beta}(\zeta-\eta)}\right] \underline{\rho}(\eta)\,d\eta \quad \bigl(\zeta \in (0,\bar{\varepsilon})\bigr),$$

with positive constants $C$, $\beta$ to be chosen. Plainly, $|Z(\zeta)|$ diverges with the same order as $P(\zeta)$ for $\zeta \to 0^+$. Hence we have:

$$\mathcal{Z}' > 0, \quad \beta \mathcal{Z}'' - C\mathcal{Z}' = -\underline{\rho} \quad \text{in } (0,\bar{\varepsilon});$$
$$\mathcal{Z}(\zeta) \to -\infty, \quad \mathcal{Z}'(\zeta) \to \infty, \quad \mathcal{Z}''(\zeta) \to -\infty \quad \text{as } \zeta \to 0.$$

Set $\tilde{Z}(x) := \mathcal{Z}(d(x,\mathcal{S}_2))$ $(x \in \mathcal{S}_2^{\bar{\varepsilon}})$. Let us show that

$$\mathcal{L}\tilde{Z}(x) \geqslant -1 \quad \text{in } \mathcal{S}_2^{\varepsilon}, \tag{5.5}$$

with a proper choice of $C$, $\beta$ and for $\varepsilon \in (0,\bar{\varepsilon}]$ possibly smaller then $\bar{\varepsilon}$.

To check (5.5) fix any $x \in \mathcal{S}_2^{\varepsilon}$ $(\varepsilon \in (0,\bar{\varepsilon}])$. As in the proof of Theorem 2.12, take new coordinates, still denoted by $x$, satisfying condition $(C)$ in Section 4.1. In particular, we have:

$$x = (0,\ldots,0,x_n), \quad x^*(x) = 0, \quad d(x,\mathcal{S}_2) = x_n, \tag{5.6}$$
$$T_0\mathcal{S}_2 = \bigl\{(x_1,\ldots,x_n) \in \mathbb{R}^n \bigm| x_n = 0\bigr\}. \tag{5.7}$$

By abuse of notation, denote again by $a_{ij}$, $b_i$ the coefficients of $\mathcal{M}$ in the new coordinates. We can choose $C$ and $\beta$ such that

$$|a_{nn}| \leqslant \beta, \qquad \bigl|\mathcal{M}d(x,\mathcal{S}_2)\bigr| \leqslant C \quad \text{in } \mathcal{S}_2^{\bar{\varepsilon}};$$

here use of (4.9)–(4.10) with $k = n-1$ has been made. Using (2.17), (4.9) and taking $\varepsilon \in (0,\bar{\varepsilon}]$ so small that $\mathcal{Z}(\zeta), \mathcal{Z}''(\zeta) < 0$ and $\mathcal{Z}'(\zeta) > 0$ for any $\zeta \in (0,\varepsilon]$, we obtain:

$$\begin{aligned}\mathcal{M}\tilde{Z}(x) &= a_{nn}(x)\mathcal{Z}''\bigl(d(x,\mathcal{S}_2)\bigr) + \bigl[\mathcal{M}d(x,\mathcal{S}_2)\bigr]\mathcal{Z}'\bigl(d(x,\mathcal{S}_2)\bigr)\\ &\geqslant \beta\mathcal{Z}''\bigl(d(x,\mathcal{S}_2)\bigr) - C\mathcal{Z}'\bigl(d(x,\mathcal{S}_2)\bigr) = -\underline{\rho}\bigl(d(x,\mathcal{S}_2)\bigr) \geqslant -\rho(x)\end{aligned}$$

for any $x \in \mathcal{S}_2^{\varepsilon}$; then inequality (5.5) follows.

In view of inequality (5.5), $\tilde{Z} - H$ with suitable $H$ is a subsolution in $\mathcal{S}_2^{\varepsilon}$ of the differential equation in (1.9). To complete the proof, we must extend its definition to $(\Omega \cup \mathcal{R}) \setminus \mathcal{S}_2^{\varepsilon}$ so as to exhibit a subsolution of problem (1.9). This is easily done arguing as in part (ii) of the proof of Theorem 2.12; we leave the details to the reader. Hence the result follows. □

**Proof of Theorem 2.18.** (i) Fix $\varepsilon \in (0, \bar{\varepsilon}]$ so small that $\varepsilon < \sigma$, with $\sigma$ as in Lemma 4.1, and $\overline{\mathcal{S}_1^\varepsilon} \cap \mathcal{R} = \emptyset$. Fix any $x \in \overline{\mathcal{S}_1^\varepsilon} \setminus \mathcal{S}_1$; as in the proof of Theorem 2.16, it is not restrictive to assume (5.6)–(5.7). In view of the assumptions about the coefficients $a_{ij}$, $b_i$, we can choose $C > 0$ such that

$$\left|\mathcal{M}d(x, \mathcal{S}_1)\right| \leqslant C \quad \text{in } \overline{\mathcal{S}_1^\varepsilon} \setminus \mathcal{S}_1. \tag{5.8}$$

Then choose $c_1 > 0$ such that

$$c_1 \leqslant \frac{1}{C} \int_{\varepsilon/2}^{\varepsilon} e^{\frac{C}{\alpha}\eta} \bar{\rho}(\eta)\, d\eta. \tag{5.9}$$

Define:

$$\mathcal{V}(\zeta) := c_1 e^{-\frac{C}{\alpha}\zeta} - \frac{1}{C} \int_{\zeta}^{\varepsilon} \left[e^{\frac{C}{\alpha}(\eta-\zeta)} - 1\right] \bar{\rho}(\eta)\, d\eta \quad \bigl(\zeta \in (0, \varepsilon)\bigr),$$

with $C, c_1$ as above, $\alpha > 0$ being the ellipticity constant in $(E_3)$. It is easily seen that

$$\alpha \mathcal{V}'' + C\mathcal{V}' = -\bar{\rho} \quad \text{in } (0, \varepsilon),$$
$$\mathcal{V}' > 0, \quad \mathcal{V}'' < 0 \quad \text{in } (0, \varepsilon/2)$$

(use of the choice (5.9) is made for the latter). Using assumption (2.20), it is also easily checked that $\mathcal{V}$ is bounded from below in $(0, \varepsilon/2)$; moreover $\mathcal{V}$ is increasing, there, hence there exists $\mathcal{V}(0) := \lim_{\zeta \to 0^+} \mathcal{V}(\zeta)$.

(ii) Set $\tilde{V}(x) := \mathcal{V}(d(x, \mathcal{S}_1)) - \mathcal{V}(0)$ $(x \in \overline{\mathcal{S}_1^{\varepsilon/2}} \setminus \mathcal{S}_1)$; let us prove that

$$\mathcal{L}\tilde{V}(x) \leqslant -1 \quad \text{in } \overline{\mathcal{S}_1^{\varepsilon/2}} \setminus \mathcal{S}_1. \tag{5.10}$$

In fact, by (5.8) we have:

$$\begin{aligned}\mathcal{M}\tilde{V}(x) &= \mathcal{V}''\bigl(d(x, \mathcal{S}_1)\bigr) \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial d(x, \mathcal{S}_1)}{\partial x_i} \frac{\partial d(x, \mathcal{S}_1)}{\partial x_j} + \bigl[\mathcal{M}d(x, \mathcal{S}_1)\bigr]\mathcal{V}'\bigl(d(x, \mathcal{S}_1)\bigr) \\ &\leqslant \alpha \mathcal{V}''\bigl(d(x, \mathcal{S}_1)\bigr) + C\mathcal{V}'\bigl(d(x, \mathcal{S}_1)\bigr) = -\bar{\rho}\bigl(d(x, \mathcal{S}_1)\bigr) \leqslant -\rho(x),\end{aligned}$$

for any $x \in \overline{\mathcal{S}_1^{\varepsilon/2}} \setminus \mathcal{S}_1$; here use of (2.21), assumption $(E_3)$ and the above properties of $\mathcal{V}$ has been made. Then inequality (5.10) follows.

(iii) Consider the solution $W \geqslant 0$ of the problem:

$$\begin{cases} \mathcal{L}W = -1 & \text{in } \Omega \setminus \overline{\mathcal{S}_1^{\varepsilon/2}}, \\ W = 1 & \text{on } \mathcal{R}, \\ W = 0 & \text{on } \partial\mathcal{S}_1^{\varepsilon/2} \setminus \mathcal{S}_1. \end{cases} \tag{5.11}$$

Define

$$V := \begin{cases} c_0 \tilde{V} & \text{in } \mathcal{S}_1^{\varepsilon/2}, \\ W + k & \text{in } \Omega \setminus \overline{\mathcal{S}_1^{\varepsilon/2}}, \end{cases}$$

where

$$c_0 := \max\left\{1, \frac{1}{\alpha \mathcal{V}'(\varepsilon/2)} \left(\sum_{i,j=1}^{n} |a_{ij}|_{L^\infty(\Omega)}\right) |\nabla W|_{L^\infty(\partial\mathcal{S}_1^{\varepsilon/2} \setminus \mathcal{S}_1)}\right\}, \quad k := c_0\bigl(\mathcal{V}(\varepsilon/2) - \mathcal{V}(0)\bigr) > 0.$$

We shall prove the following:

**Claim.** *The function $V$ is a positive supersolution of problem* (1.5) *satisfying condition* (1.6).

In view of Theorem 1.1, from the above claim the conclusion follows.
To prove the claim we must show the following:

(a) $V \in C(\Omega \cup \mathcal{R})$;
(b) $\int_\Omega V\mathcal{M}^*\psi\,dx \leqslant -\int_\Omega \rho\psi\,dx$ for any $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$;
(c) condition (1.6) is satisfied.

To check (a), observe that the condition

$$c_0\tilde{V} = W + k \quad \text{on } \partial\mathcal{S}_1^{\varepsilon/2} \setminus \mathcal{S}_1,$$

reads

$$c_0\big(\mathcal{V}(\varepsilon/2) - \mathcal{V}(0)\big) = k,$$

thus it follows from the choice of $k$. The continuity of $V$ elsewhere is clear from its very definition; hence (a) follows.
Concerning (b), an easy calculation gives:

$$\int_\Omega V\mathcal{M}^*\psi\,dx \leqslant -\int_\Omega \rho\psi\,dx - \int_\Gamma \sum_{i,j=1}^n a_{ij}(x)\frac{\partial}{\partial x_i}(c_0\tilde{V} - W)\nu_j(x)\,dx,$$

for any $\psi \in C_0^\infty(\Omega)$, $\psi \geqslant 0$; here $\Gamma := \partial\mathcal{S}_1^{\varepsilon/2} \cap \operatorname{supp}\psi$ and $\nu(x) \equiv (\nu_1(x), \dots, \nu_n(x))$ denotes the outer normal to $\mathcal{S}_1^{\varepsilon/2}$ at $x \in \partial\mathcal{S}_1^{\varepsilon/2} \setminus \mathcal{S}_1$. Since

$$\sum_{i,j=1}^n a_{ij}(x)\frac{\partial\tilde{V}}{\partial x_i}\nu_j(x) = \mathcal{V}'(\varepsilon/2)\sum_{i,j=1}^n a_{ij}(x)\nu_i(x)\nu_j(x) \geqslant \alpha\mathcal{V}'(\varepsilon/2) > 0,$$

inequality (b) follows from the above choice of $c_0$.
Finally, concerning (c) observe that

$$\inf_{\Omega\cup\mathcal{R}} V = c_0 \inf_{\mathcal{S}_1^{\varepsilon/2}} \tilde{V} = 0 < 1 + k = \inf_{\mathcal{R}}(W + k) = \inf_{\mathcal{R}} V;$$

hence (1.6) is satisfied. This completes the proof. □

**Proof of Theorem 2.21.** By Proposition 2.7(i) $\mathcal{S}_1$ is attracting. Then, in view of Theorem 2.5, there exists a solution of problem (1.7) satisfying (2.5). Since the constant in (2.5) is arbitrary and $\overline{\mathcal{R}} \cap \overline{\mathcal{S}_1} = \emptyset$ (see assumption $(A_1)$(ii)), the conclusion follows. □

**Proof of Theorem 2.23.** By Proposition 2.9 the function $Z$ defined in (2.10) is a subsolution of problem (1.9). In view of Theorem 1.2, the conclusion follows. □

It remains to prove Propositions 2.7–2.9. To this purpose we need the following lemma (see [4] for the proof).

**Lemma 5.1.** *Let assumptions $(F_1)$–$(F_3)$ be satisfied. Suppose that $\sigma_{ij} \in C^1(\overline{\Sigma^\varepsilon})$ for some $\varepsilon > 0$ $(i, j, = 1 \dots, n)$, and let $\Sigma \subseteq \Sigma_1 \cup \Sigma_2$ be a smooth connected component of $\partial\Omega$. Then,*

(i) *for any $x \in \Sigma$ there holds,*

$$\frac{\partial d(x,\Sigma)}{\partial x_i} = -\nu_i(x), \tag{5.12}$$

$$\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 d(x,\Sigma)}{\partial x_i\,\partial x_j} = \sum_{i,j=1}^n \frac{\partial a_{ij}(x)}{\partial x_j}\nu_i(x); \tag{5.13}$$

(ii) *there exist* $\varepsilon > 0$, $C > 0$ *such that, for any* $x \in \Sigma^\varepsilon$,

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial d(x,\Sigma)}{\partial x_i} \frac{\partial d(x,\Sigma)}{\partial x_j} \leqslant C\big[d(x,\Sigma)\big]^2; \tag{5.14}$$

*moreover, if* $\Sigma \subseteq \Sigma_1$, *there holds,*

$$\mathcal{M} d(x,\Sigma) \geqslant -C d(x,\Sigma). \tag{5.15}$$

**Proof of Proposition 2.7.** (i) Since $\Sigma \subseteq \Sigma_2$ is compact, there holds:

$$C := \min_{x \in \Sigma} \frac{\beta_F(x)}{\rho(x)} > 0.$$

Hence from (5.12)–(5.13) we obtain:

$$\mathcal{L} d(x,\Sigma) = -\frac{1}{\rho(x)} \sum_{i=1}^{n} \left[ b_i(x) - \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \right] \nu_i(x) = -\frac{\beta_F(x)}{\rho(x)} < -C,$$

for any $x \in \Sigma$. By continuity we get:

$$\mathcal{L} d(x,\Sigma) \leqslant -C \quad \text{in } \Sigma^\varepsilon, \tag{5.16}$$

with $\varepsilon > 0$ suitably small. Choosing $V := \frac{d(\cdot,\Sigma)}{C}$ in Definition 2.3, the claim follows.

(ii) Let $x^0 \in \Sigma$; fix $r > 0$ so small that $\overline{\Omega \cap B_r(x^0)} \subseteq \Sigma^\varepsilon$. Define:

$$h(x) := \lambda_1\big[d(x,\Sigma) + \lambda_2 |x - x^0|^2\big] \quad \big(x \in \overline{\Omega \cap B_r(x^0)};\ \lambda_1, \lambda_2 > 0\big).$$

It is easily seen that

$$\mathcal{L} h(x) = \lambda_1 \left\{ \mathcal{L} d(x,\Sigma) + \frac{2\lambda_2}{\rho(x)} \left[ \sum_{i=1}^{n} \big(a_{ii}(x) + b_i(x)\big(x_i - x_i^0\big)\big) \right] \right\}, \tag{5.17}$$

for any $x \in \Omega \cap B_r(x^0)$. In view of (5.16) and the boundedness of the coefficients, by a proper choice of $\lambda_1, \lambda_2$ the conclusion follows. □

**Proof of Proposition 2.9.** Take $\varepsilon \in (0, \varepsilon_0)$ as in Lemma 5.1(ii) and such that $\overline{\Sigma^\varepsilon} \subseteq \Omega \cup \Sigma$. Then it is easily seen that for any $x \in \Sigma^{\varepsilon/2}$,

$$\begin{aligned}
\mathcal{L} Z(x) &= \frac{[d(x,\Sigma)]^{-\alpha}}{\rho(x)} \left\{ \alpha\big[d(x,\Sigma)\big]^{-1} \mathcal{M} d(x,\Sigma) - \alpha(\alpha+1)\big[d(x,\Sigma)\big]^{-2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial d(x,\Sigma)}{\partial x_i} \frac{\partial d(x,\Sigma)}{\partial x_j} \right\} \\
&\geqslant \frac{[d(x,\Sigma)]^{-\alpha}}{\rho(x)} \big\{ -\alpha C - \alpha(\alpha+1) C \big\} \geqslant -C_1 \big[d(x,\Sigma)\big]^{-\alpha},
\end{aligned}$$

where $C_1 = C_1(\alpha) := [\alpha(\alpha+2)C] \big/ \min_{\overline{\Omega}} \rho$; here use of (5.14)–(5.15) has been made. Then

$$\mathcal{L} Z \geqslant C_1(Z + H) \geqslant C_1 Z \quad \text{in } \Sigma^{\varepsilon/2} \tag{5.18}$$

with $\varepsilon > 0$ suitably small (see (2.9)–(2.10)). On the other hand, there holds:

$$\mathcal{L} Z = 0 \quad \text{in } \Omega \setminus \Sigma^\varepsilon, \tag{5.19}$$

$$\mathcal{L} Z(x) \geqslant -C_2 \quad \text{in } \Sigma^\varepsilon \setminus \Sigma^{\varepsilon/2}, \tag{5.20}$$

for some $C_2 > 0$. Choosing $H \geqslant C_2/c_0$ and $\alpha > 0$ so small that $C_1 \leqslant c_0$, from (5.18)–(5.20) we obtain:

$$\mathcal{L} Z \geqslant c_0 Z \quad \text{in } \Omega;$$

then the conclusion follows. □

## 6. Examples

In this section we give a few applications of the above results, limiting ourselves to the elliptic case. We always assume $\phi \in C(\Omega) \cap L^\infty(\Omega)$, $\gamma \in C(\mathcal{R})$.

(a) *An application of Theorem* 2.12. Consider the problem:

$$\begin{cases} (x^2+d^2)u_{xx} + 2xyu_{xy} + (y^2+d^2)u_{yy} + u_{zz} - u = \phi & \text{in } \Omega, \\ u = \gamma & \text{on } \mathcal{R}, \end{cases} \tag{6.1}$$

where $\Omega := B_2(0) \setminus \{(x,y,0) \mid x^2+y^2=1\} \subseteq \mathbb{R}^3$, $\mathcal{R} = \partial B_2(0)$, $\mathcal{S} = \{(x,y,0) \mid x^2+y^2=1\}$, $d \equiv d((x,y,z),\mathcal{S})$.

Here $\dim \mathcal{S} = n-2$; moreover, the diffusion matrix of the operator considered in (6.1) is elliptic in $\Omega \cup \mathcal{R}$, but has points of degeneracy on $\mathcal{S}$.

It is easily seen that in this case,

$$M(x,y,z) = \begin{pmatrix} -y & x & 0 \\ -x & -y & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_\perp(x,y,z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \big((x,y,z) \in \mathcal{S}\big).$$

By Theorems 2.5 (with $\mathcal{S}_1 = \emptyset$, $F \equiv 1$), 2.12 and Remark 4.2, problem (6.1) is well posed in $L^\infty(\Omega)$.

(b) *The limiting value* $\beta = 1$ *in condition* $(A_2)(\mathrm{ii})'$ *of Theorem* 2.12 *is not allowed.* Consider the problem:

$$\begin{cases} \Delta u - \frac{4}{x^2+y^2}(xu_x + yu_y) - u = \phi & \text{in } \Omega, \\ u = \gamma & \text{on } \mathcal{R}, \end{cases} \tag{6.2}$$

where $\Omega = B_1(0) \setminus \{0\} \subseteq \mathbb{R}^2$, $\mathcal{R} = \partial B_1(0)$, $\mathcal{S} = \{0\}$.

The function $V(x,y) := x^2 + y^2$ is a supersolution of equation:

$$\mathcal{L}V = -1 \quad \text{in } \Omega,$$

moreover, it satisfies

$$0 = \inf_{\Omega \cup \mathcal{R}} V < \inf_{\mathcal{R}} V = 1.$$

By Theorem 1.1, problem (6.2) has infinitely many bounded solutions.

Observe that $\dim \mathcal{S} = 0 = n-2$ and $r(0) = 2$, but Theorem 2.12 does not apply. In fact, condition $(A_2)(\mathrm{ii})'$ is not satisfied, since

$$\big|b(x,y)\big| = \frac{4}{\sqrt{x^2+y^2}} \quad \big((x,y) \in \Omega\big).$$

Similar remarks can be made when $r(y) \geqslant 3$ or $r(y) \geqslant 4$ for any $y \in \mathcal{S}$; we omit the details.

(c) *A comparison between the results of Theorems* 2.16*,* 2.20 *and those of Theorems* 2.21*,* 2.23. Consider the problem:

$$\begin{cases} d^\alpha \Delta u - u = \phi & \text{in } \Omega, \\ u = \gamma & \text{on } \mathcal{R}, \end{cases} \tag{6.3}$$

where $\alpha \in \mathbb{R}, \alpha \neq 0$, $\Omega = B_2(0) \setminus \overline{B}_1(0) \subseteq \mathbb{R}^3$, $\mathcal{R} = \partial B_2(0)$, $\mathcal{S} = \partial B_1(0)$, $d \equiv d(x) \equiv d(x,\mathcal{S})$ $(x \in \Omega \cup \mathcal{R})$. Depending on the values of the parameter $\alpha \neq 0$ different situations arise, as discussed below.

*Nonuniqueness case:* $\alpha < 2$. Set $\rho(x) := [d(x)]^{-\alpha}$ $(x \in \Omega \cup \mathcal{R})$. By Theorem 2.20, problem (6.3) admits infinitely many bounded solutions in $L^\infty(\Omega)$.

It is worth observing that Theorems 2.21 and 2.23 do not apply in this case, due to the lack of regularity. In fact, problem (6.3) can be regarded as a particular case of problem (1.7) with

$$\mathcal{S} = \mathcal{S}_1, \quad \rho = c \equiv 1, \quad a_{ij} = d^\alpha \delta_{ij}, \quad b_i \equiv 0 \quad (i,j = 1,2,3).$$

When $\alpha < 1$ condition $(F_2)(\mathrm{ii})$ is not satisfied; hence the functions $\alpha_F$ and $\beta_F$ are not well defined and Theorem 2.21 cannot be applied. However, its conclusion holds true, as seen above.

When $\alpha = 1$ conditions $(F_1)$–$(F_3)$ are satisfied; moreover, it is easily seen that $\mathcal{S}_1 \subseteq \Sigma_2$. However, Theorem 2.21 cannot be applied, since $\sigma_{ij} = d^{1/2}\delta_{ij} \notin C^1(\overline{\mathcal{S}_1^\varepsilon})$ (although its conclusion holds true, as already remarked).

When $\alpha \in (1, 2)$ condition $(F_2)$(ii) is not satisfied, yet in this case the functions $\alpha_F$ and $\beta_F$ are well defined and $\mathcal{S}_1 \subseteq \Sigma_1$. Theorem 2.23 cannot be applied (observe that moreover $\sigma_{ij} = d^{\alpha/2}\delta_{ij} \notin C^1(\overline{\mathcal{S}_1^\varepsilon})$); in fact, as already seen, its conclusion is false in this case.

*Uniqueness case:* $\alpha \geqslant 2$. Set $\rho(x) := [d(x)]^{-\alpha}$ $(x \in \Omega \cup \mathcal{R})$. In view of Theorem 2.16, there exists at most one bounded solution of problem (6.3).

Observe that problem (6.3) is a particular case of problem (1.7) with

$$\mathcal{S} = \mathcal{S}_2, \quad \rho = c \equiv 1, \quad a_{ij} = d^\alpha \delta_{ij}, \quad b_i \equiv 0 \quad (i, j = 1, 2, 3).$$

In this case conditions $(F_1)$–$(F_3)$ are satisfied; moreover, it is easily seen that $\mathcal{S}_2 \subseteq \Sigma_1$. By Theorem 2.23 there exists at most one bounded solution of problem (6.3), in agreement with the above conclusion obtained by Theorem 2.16.

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